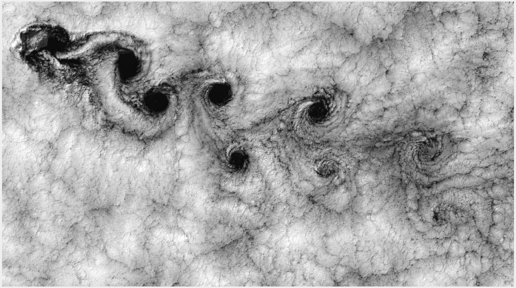
Two-Dimensional Turbulence:

order out of a chaos

Max Brazhnikov

Institute of
Solid State Physics,
Chernogolovka

November 17, 2022



Get Notification About Future Events

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla)\vec{v} = -\frac{\nabla p}{\rho} + \nu \Delta \vec{v}$$

$$\frac{\partial \rho}{\partial t} + \texttt{div}(\rho \vec{v}) = 0$$

Reynolds number

$$(v, \nabla)v \sim U^2/L$$

$$\nu \Delta v \sim \nu U/L^2$$

$$\frac{(v, \nabla)v}{\nu \Delta v} \sim \frac{UL}{\nu} \equiv Re$$

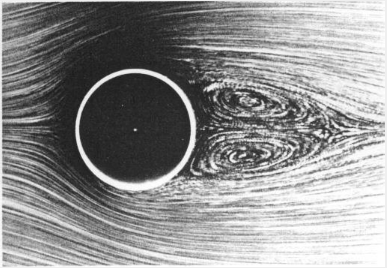
Dimensionless variables

$$(x, y, z) = (x, y, z)/L$$

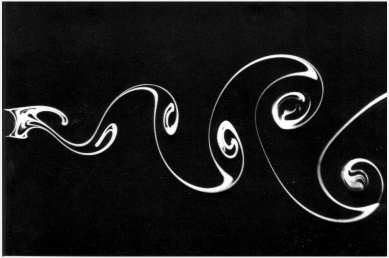
$$v = \vec{v}/U, \quad p = p/\rho U^2$$

$$\nu = \frac{\nu}{UL} = Re^{-1}$$

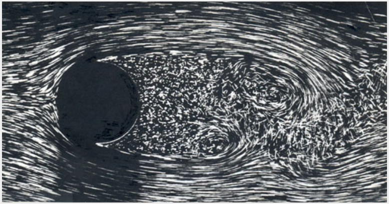
$$\frac{\partial v}{\partial t} + (v, \nabla)v = -\nabla p + Re^{-1} \Delta v$$



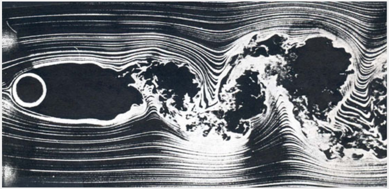
$Re = 26$



$Re = 140$

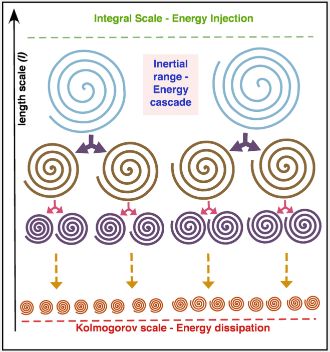


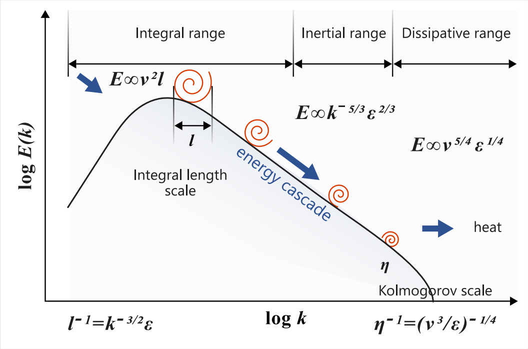
$Re = 2000$



$Re = 10^4$

Homogeneous Isotropic Turbulence in 3D





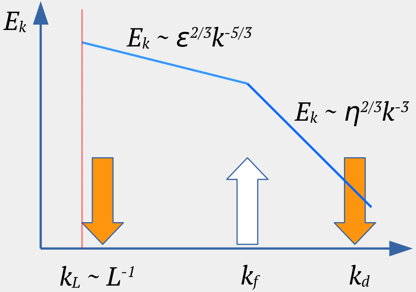
$$L/l_d \sim Re^{3/4}$$

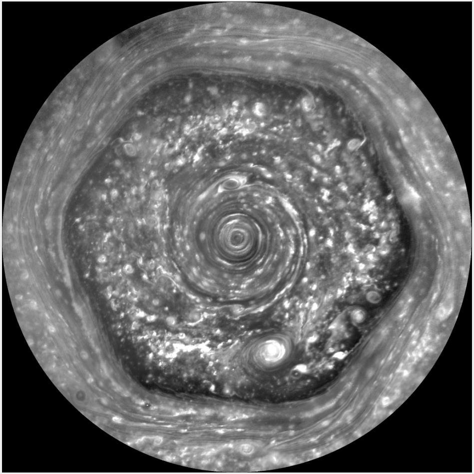
Airplane at 800 km/h: $Re \sim 10^9$ $\quad l_d \sim 0.01\, mm$ $\quad \Delta \sim 10^{-7}\, s$

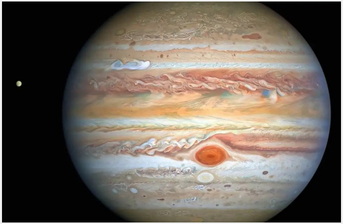
$$\vec{v}_{(3)} = (\vec{v}_{(2)}(z), 0)$$
$$\vec{v}_{(2)}|_{z=0} = 0, \qquad \vec{v}_{(2)}|_{z=h} = \vec{v}$$

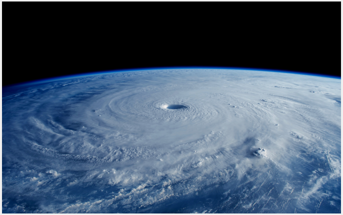
$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla)\vec{v} = -\frac{\nabla p}{\rho} - \alpha\vec{v} + \nu\Delta\vec{v} + \vec{f}$$ (2d Navier–Stokes eq.)

$$E = \frac{1}{2S}\int v^2 \mathrm{d}S$$
$$E_k = \frac{1}{2}\sum_{|\mathbf{k}|=k} |\vec{v}_{\mathbf{k}}|^2$$
$$\gamma_k = \alpha + \nu k^2$$
$$L_\alpha \sim \epsilon^{1/2}\alpha^{-3/2},\ U_L \sim \epsilon^{1/2}\alpha^{-1/2}$$
$$\epsilon(t) = S^{-1}\int \vec{f}\cdot\vec{v}(t)\,\mathrm{d}x\,\mathrm{d}y$$









Credit: NASA, ESA, A. Simon (Goddard Space Flight Center), and M. H. Wong (University of California, Berkeley) and the OPAL team.

Experimental Setup

$f_x = 0$

$f_y = f_0 \cos\frac{\pi x}{a} \sin\frac{\pi y}{a}$

$a = 1\,cm$

$k_f = 4.4\,cm^{-1}$

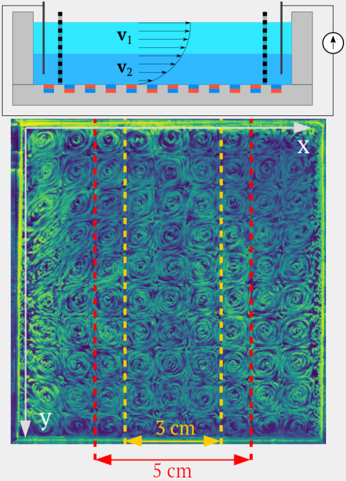
$B(z) = \frac{B_0\ R^2}{(R^2+z^2)^{3/2}}$

$B_0 \approx 1.1\,T,\ 2R = 0.5\,cm$

$j = \frac{0.5\ A}{10\,cm\times 5\,mm} = 0.1\ A/cm^2$

$f_0 = \frac{j\,B}{\rho} = \frac{10^3\ A/m^2\times 0.1\,T}{1.1\times 10^3\ kg/m^3}$

$f_0 \sim 10\,cm/s^2$



$$\epsilon(t) = S^{-1}\int \cos(k_f x)\sin(k_f y)\,v_y(t,x,y)\,\mathrm{d}x\,\mathrm{d}y > 0 \Rightarrow$$

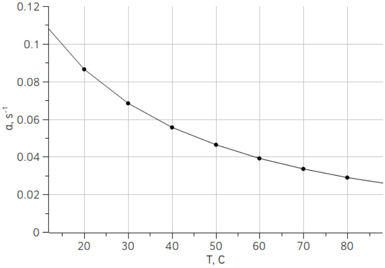
$$\Rightarrow \langle v_{\pm k_f}\rangle \neq 0,\ \langle v(t,x,y)\rangle \neq 0$$

Two Fluid Layers: fluorocarbon and Electrolyte

Cell dimensions $L_x \times L_y$: 10×10 cm, 5×10 cm and 3×10 cm

Upper: $H_2O + 15\%\, KNO_3$
$h = 0.5 - 0.7\, cm$,
$\rho = 1.11\, g/cm^3$,
$\nu \simeq 0.0085\, cm^2/s$

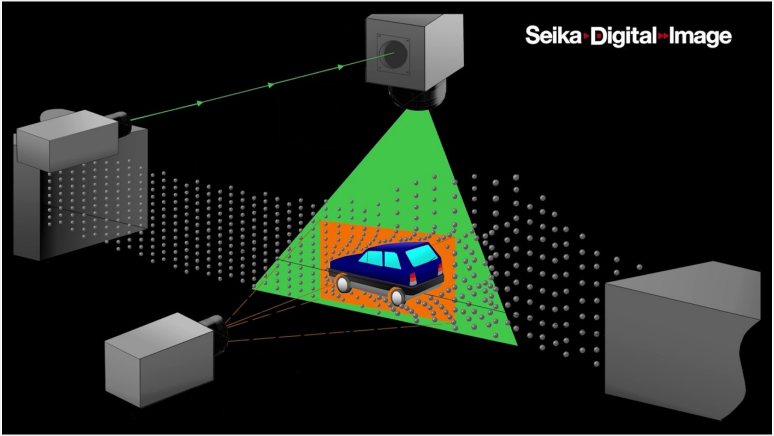
Lower: $C_{10}F_{18}$
$h = 0.3\, cm$,
$\rho = 1.98\, g/cm^3$,
$\nu = 0.028\, cm^2/s$



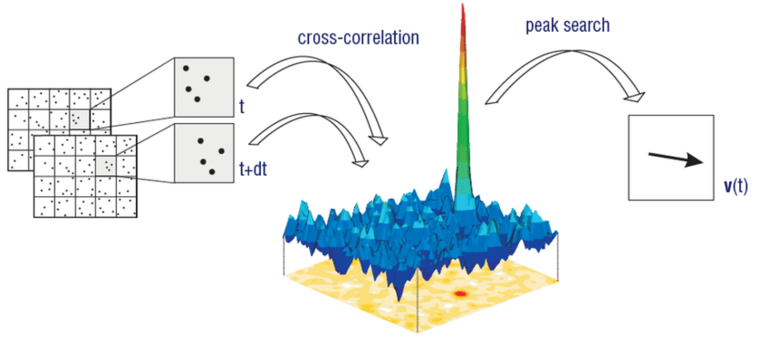
$$\tan\sqrt{\frac{\alpha}{\nu_t}}h_t \,\tan\sqrt{\frac{\alpha}{\nu_b}}h_b = \frac{\rho_b}{\rho_t}\sqrt{\frac{\nu_b}{\nu_t}}$$

FC ($3\, mm$) & Electrolyte ($5\, mm$) — $\alpha = 0.068\, s^{-1}$, $\tau = 7.4\, s$
Electrolyte ($8\, mm$) — $\alpha = 0.033\, s^{-1}$, $\tau = 15.3\, s$
FC ($3\, mm$) & Electrolyte ($7\, mm$) — $\alpha = 0.037\, s^{-1}$, $\tau = 13.6\, s$

Particle Image Velocimetry



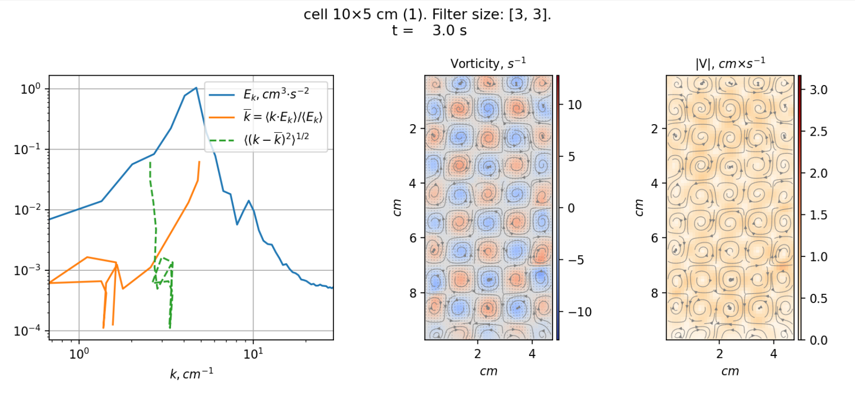
Credit: SEIKA Digital Image Corporation



B. Wieneke, PIV Uncertainty Quantification and Beyond,
`https://doi.org/10.13140/RG.2.2.26244.42886`

The Show Must Go On

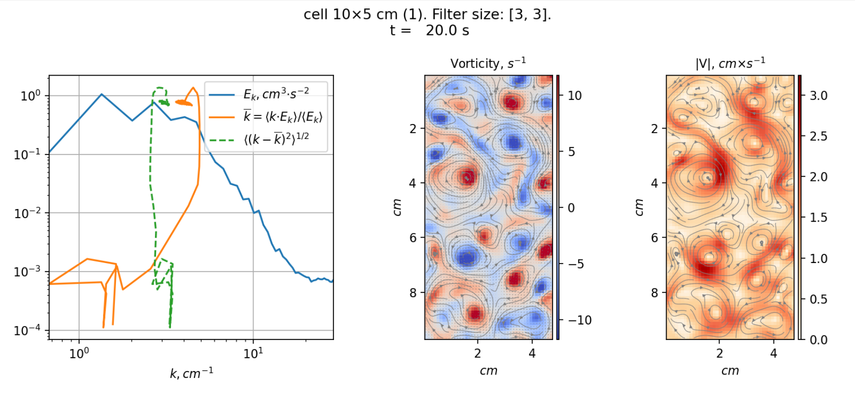
FC&Electrolyte. Cell: 5×10 cm. $t - t_{on} = 1\,s$



$$t - t_{on} = 1\,s$$

$$\bar{k}(t) = \langle k \rangle \qquad \Delta k(t) = \langle (k - \bar{k}) \rangle^{1/2} \qquad E(t)$$

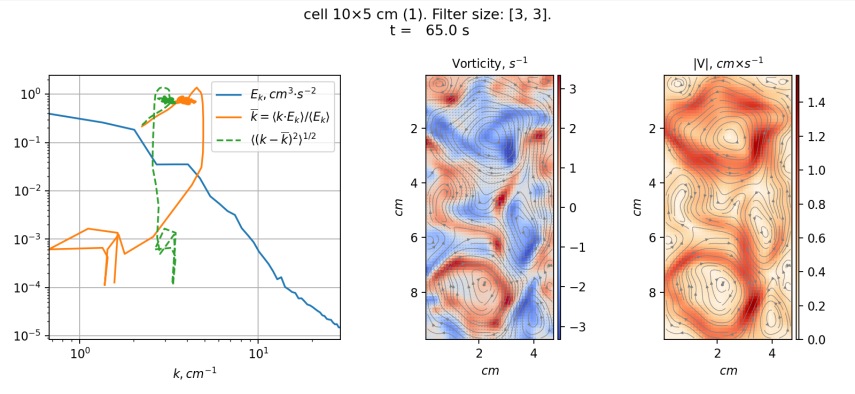
$$\langle \xi \rangle = \frac{\int \xi E_k(t) \mathrm{d}k}{\int E_k(t) \mathrm{d}k}$$



$$t - t_{on} = 18\,s$$

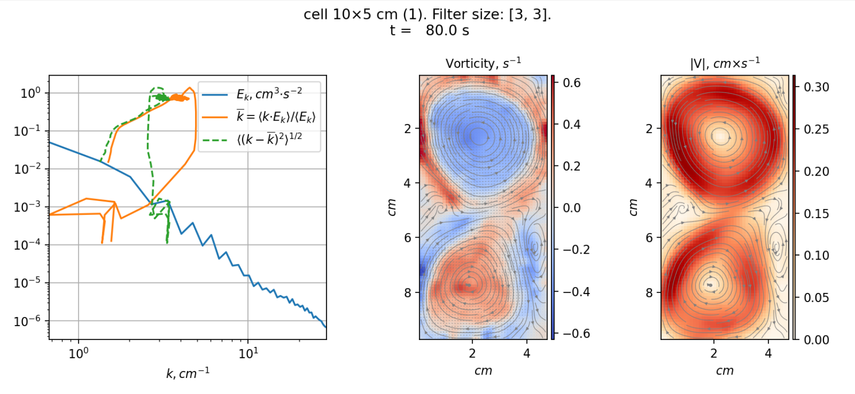
$$Re(k_f) \simeq 300,\ Re(k_L) \simeq 10^3$$

FC&Electrolyte. Cell: 5×10 cm. $t - t_{off} = 3\,s$



$$t - t_{off} = 3\,s$$

FC&Electrolyte. Cell: 5×10 cm. $t - t_{off} = 18\, s$



$$t - t_{off} = 18\, s$$

$$k_L = 2\pi / L_x \approx 1.3\, cm^{-1}$$

$$\bar{k} \approx 1.6\, cm^{-1}$$

Space Filtering for Velocity Decomposition

$$v(\mathbf{r},t) = \bar{v}(\mathbf{r},t) + \tilde{v}(\mathbf{r},t)$$

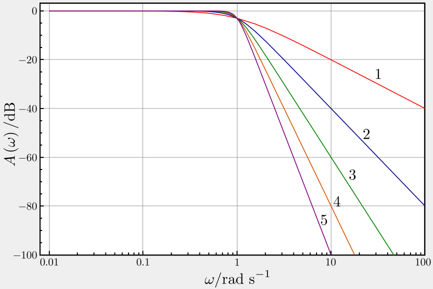
$$v_{\mathbf{k}}(t) = \sum_{\mathbf{r}} v(\mathbf{r},t) e^{i(\mathbf{k},\mathbf{r})}$$

$$\bar{v}(\mathbf{r},t) = \sum_{\mathbf{k}} v_{\mathbf{k}}(t) G(\mathbf{k}) e^{-i(\mathbf{k},\mathbf{r})}$$

$$G^2(k) = \frac{1}{1+\left(\frac{k}{k_c}\right)^{2n}}$$

$$\tilde{v}(\mathbf{r},t) = v(\mathbf{r},t) - \bar{v}(\mathbf{r},t)$$

Butterworth low-pass filter[1]



$$k_c = 2.1\, cm^{-1}$$

[1]https://en.wikipedia.org/wiki/Butterworth_filter

Space Filtering & Energy Flux

$$\overline{\frac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla)\vec{v}} = \overline{-\frac{\nabla p}{\rho} - \alpha\vec{v} + \nu\Delta\vec{v} + \vec{f}} \qquad \mid \times\bar{v}_i,\ \int dS$$

$$\frac{\partial E}{\partial t} = -\Pi - 2\gamma E + \Gamma$$

$E(k) = (2S)^{-1} \int \bar{v}^2\, dS$ **!**

$\gamma(k) = \alpha + \nu\langle k\rangle^2$

$\Gamma(k) = S^{-1} \int \bar{f}_i\bar{v}_i\, dS$

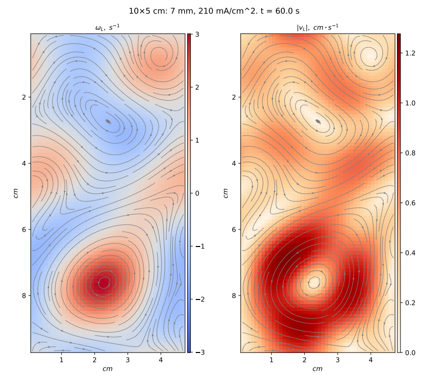
$\Pi(k) = -S^{-1} \int \tilde{v}_i\tilde{v}_j \frac{d\bar{v}_i}{dx_j}\, dS$

Direct cascade: $\Pi(k) > 0$, inverse cascade: $\Pi(k) < 0$.

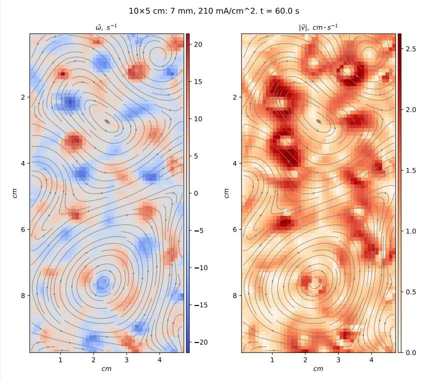
Steady state averaged over time ($k_{filter} \ll k_{force}$):

$$\frac{\partial E}{\partial t} \simeq 0,\ \Gamma \simeq 0 \quad \Rightarrow \quad \Pi + 2\gamma E \simeq 0, \quad \frac{\Pi}{E} \simeq -2\gamma$$

Coherent Flow. Cell: 5×10 cm. $t - t_{on} = 60\ s$



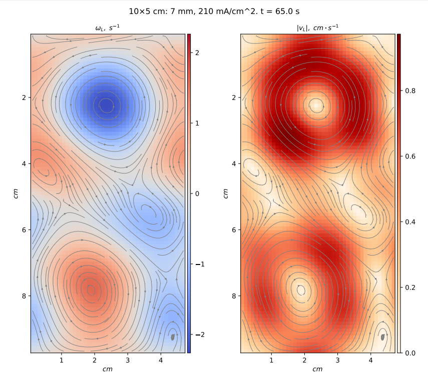
$$\bar{v}(\mathbf{r}, t)$$



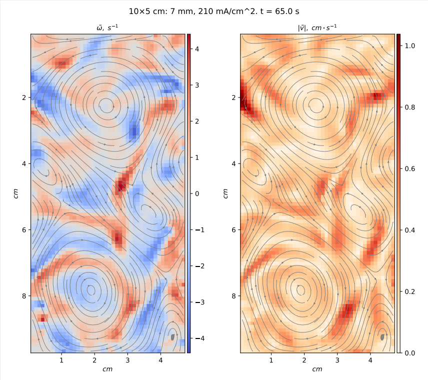
$$\tilde{v}(\mathbf{r}, t)$$

$$v(\mathbf{r}, t) = \bar{v}(\mathbf{r}, t) + \tilde{v}(\mathbf{r}, t)$$

Coherent Flow. Cell: 5×10 cm. $t - t_{off} = 3\,s$



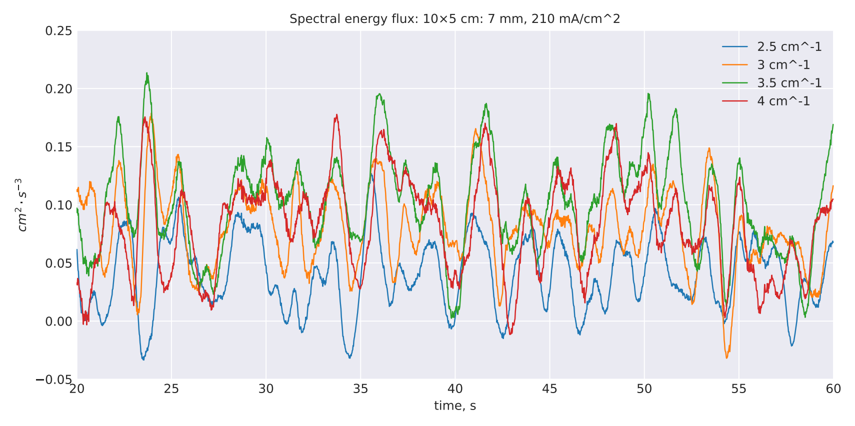
$\bar{v}(\mathbf{r}, t)$



$\tilde{v}(\mathbf{r}, t)$

$$\Pi = -\tilde{v}_i \tilde{v}_j \frac{d\bar{v}_i}{dx_j}$$

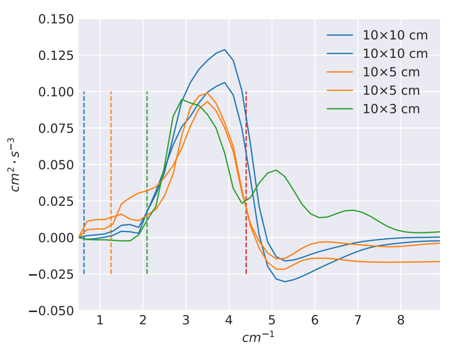
Energy Flux. Fluctuations



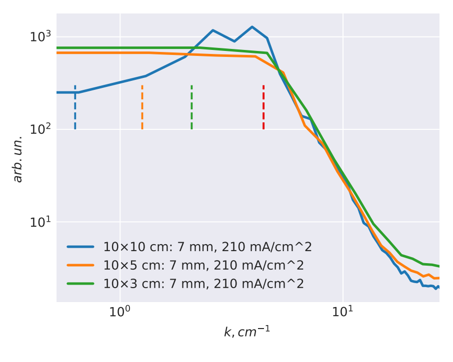
$$-\Pi(k,t)$$

Energy Flux. $L_x = [3, 5, 10]\, cm$

$-\Pi(k)$



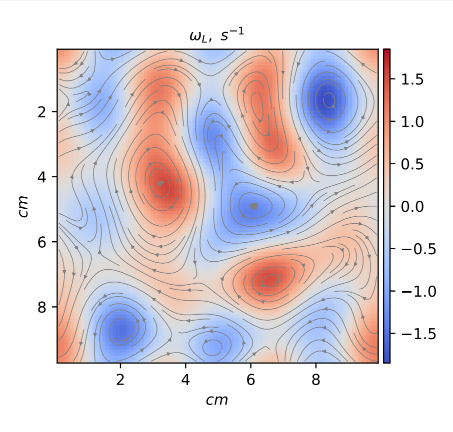
$E(k)$

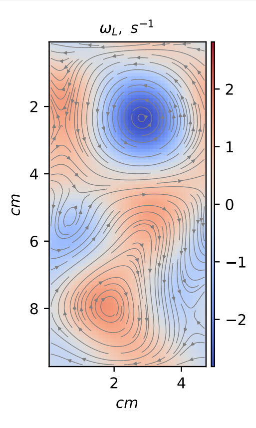


$$\Pi = 0.1\, cm^2 \cdot s^{-3}$$
$$E_c = 0.3\, cm^2 \cdot s^{-2}$$
$$\gamma = 0.077\, s^{-1}$$

$$L_c \sim \Pi^{1/2} \gamma^{-3/2} = 14.8\, cm > L_x$$
$$\Pi / E = 0.3\, s^{-1} > 2\gamma = 0.15\, s^{-1}$$

Large-Scale Flow in Square and Rectangle





$L_x = 10\,cm$ $\qquad$ $L_x = 5\,cm$

Vorticity of the large-scale flow.