Topological media:
quantum liquids, topological insulators & quantum vacuum

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1. Introduction: Universe as object of ultralow temperature physics

- * quantum vacuum as topological medium
- * effective quantum field theories emerging at low T

2. Fermi surface as topological object

- *normal 3He, metals,

3. Dirac (Fermi) points as topological objects

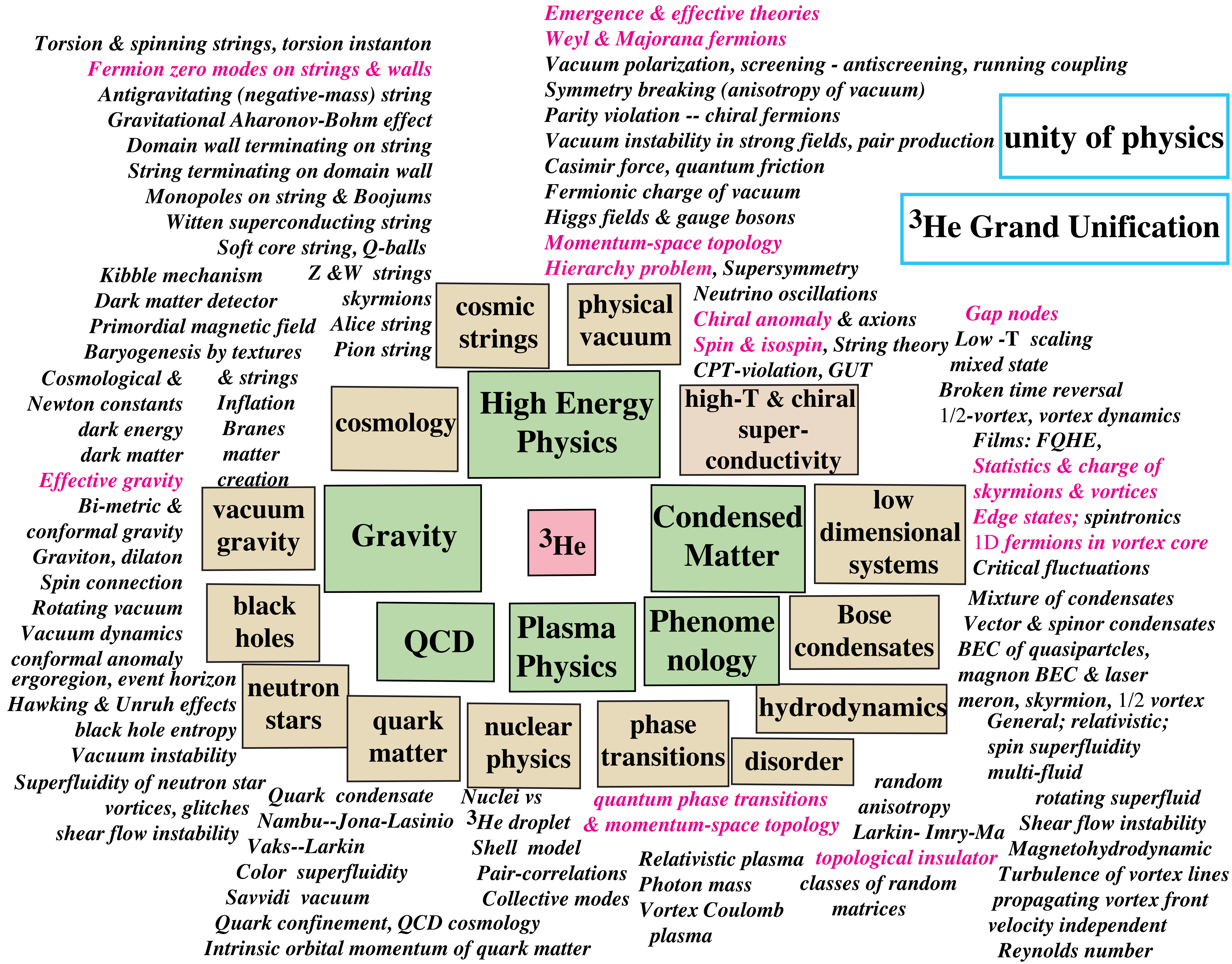
- * superfluid 3He-A, semimetals, cuprate superconductors, graphene, vacuum of Standard Model of particle physics in massless phase
- * topological invariants for gapless 2D and 3D topological matter
- * QED, QCD and gravity as emergent phenomena
- * quantum vacuum as cryo-crystal, α-phase of superfluid 3He

4. Fully gapped topological media

- * superfluid 3He-B, topological insulators, chiral superconductors, vacuum of Standard Model of particle physics in present massive phase
- * topological invariants for gapped 2D and 3D topological matter
- * edge states & Majorana fermions (planar phase of 3He & surface of 3He-B)
- * intrinsic QHE & spin-QHE

5. Conclusion

- * important role of momentum-space topology


unity of physics
^{3}He Grand Unification
Emergence & effective theories
Weyl & Majorana fermions
Vacuum polarization, screening - antiscreening, running coupling
Symmetry breaking (anisotropy of vacuum)
Parity violation -- chiral fermions
Vacuum instability in strong fields, pair production
Casimir force, quantum friction
Fermionic charge of vacuum
Higgs fields & gauge bosons
Momentum-space topology
Hierarchy problem, Supersymmetry
Neutrino oscillations
Chiral anomaly & axions
Spin & isospin, String theory
CPT-violation, GUT
Torsion & spinning strings, torsion instanton
Fermion zero modes on strings & walls
Antigravitating (negative-mass) string
Gravitational Aharonov-Bohm effect
Domain wall terminating on string
String terminating on domain wall
Monopoles on string & Boojums
Witten superconducting string
Soft core string, Q-balls
Kibble mechanism
Z &W strings
Dark matter detector
skyrmions
Primordial magnetic field
Alice string
Baryogenesis by textures
Pion string
Cosmological & Newton constants
dark energy
dark matter
& strings
Inflation
Branes
matter creation
Effective gravity
Bi-metric & conformal gravity
Graviton, dilaton
Spin connection
Rotating vacuum
Vacuum dynamics
conformal anomaly
ergoregion, event horizon
Hawking & Unruh effects
black hole entropy
Vacuum instability
Superfluidity of neutron star
vortices, glitches
shear flow instability
cosmic strings
physical vacuum
cosmology
High Energy Physics
high-T & chiral super-conductivity
vacuum gravity
Gravity
^{3}He
Condensed Matter
low dimensional systems
black holes
QCD
Plasma Physics
Phenome nology
Bose condensates
neutron stars
quark matter
nuclear physics
phase transitions
hydrodynamics
disorder
Gap nodes
Low -T scaling
mixed state
Broken time reversal
1/2-vortex, vortex dynamics
Films: FQHE,
Statistics & charge of skyrmions & vortices
Edge states; spintronics
1D fermions in vortex core
Critical fluctuations
Mixture of condensates
Vector & spinor condensates
BEC of quasipartcles,
magnon BEC & laser
meron, skyrmion, 1/2 vortex
General; relativistic;
spin superfluidity
multi-fluid
rotating superfluid
Shear flow instability
Magnetohydrodynamic
Turbulence of vortex lines
propagating vortex front
velocity independent
Reynolds number
Quark condensate
Nambu--Jona-Lasinio
Vaks--Larkin
Color superfluidity
Savvidi vacuum
Quark confinement, QCD cosmology
Intrinsic orbital momentum of quark matter
Nuclei vs ^{3}He droplet
Shell model
Pair-correlations
Collective modes
quantum phase transitions & momentum-space topology
Relativistic plasma
Photon mass
Vortex Coulomb plasma
random anisotropy
Larkin- Imry-Ma
topological insulator
classes of random matrices

3+1 sources of effective Quantum Field Theories in many-body system & quantum vacuum

Lev Landau

I think it is safe to say that no one understands **Quantum Mechanics**

Richard Feynman

Thermodynamics is the only physical theory of universal content

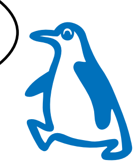
Albert Einstein

Symmetry: conservation laws, translational invariance, spontaneously broken symmetry, Grand Unification, ...

effective theories of quantum liquids: two-fluid hydrodynamics of superfluid ^{4}He & Fermi liquid theory of liquid ^{3}He

Topology: you can't comb the hair on a ball smooth, anti-Grand-Unification

missing ingredient in Landau theories



Landau view on a many body system

many body systems are simple at low energy & temperature

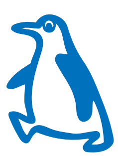
weakly excited state of liquid can be considered as system of "elementary excitations"

equally applied to:
superfluids,
solids,
&
relativistic quantum vacuum

Landau, 1941

helium liquids

ground state + elementary excitations

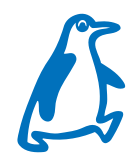


ground state = vacuum

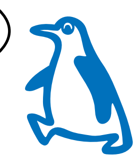
quasiparticles = elementary particles

Universe

vacuum + elementary particles



why is low energy physics applicable to our vacuum ?



characteristic high-energy scale in our vacuum is Planck energy

$$E_P = (hc^5/G)^{1/2} \sim 10^{19}\ \mathrm{GeV} \sim 10^{32}\mathrm{K}$$

high-energy physics is extremely ultra-low energy physics

highest energy in accelerators

$$E_{ew} \sim 1\ \mathrm{TeV} \sim 10^{16}\mathrm{K}$$

$$E_{ew} \sim 10^{-16} E_{Planck}$$

high-energy physics & cosmology belong to ultra-low temperature physics

T of cosmic background radiation

$$T_{CMBR} \sim 1\ \mathrm{K}$$

$$T_{CMBR} \sim 10^{-32} E_{Planck}$$

cosmology is extremely ultra-low frequency physics

cosmological expansion

$$v(r,t) = H(t)\, r$$

Hubble law

$$H \sim 10^{-60} E_{Planck}$$

Hubble parameter

our Universe is extremely close to equilibrium ground state

We should study general properties of equilibrium ground states - quantum vacua

Why no freezing at low T?

natural masses of elementary particles
are of order of characteristic energy scale
the Planck energy

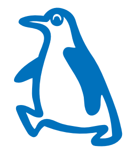
$$m \sim E_{\text{Planck}} \sim 10^{19}\ \text{GeV} \sim 10^{32}\text{K}$$

even at highest temperature
we can reach

$$T \sim 1\ \text{TeV} \sim 10^{16}\text{K}$$

everything should be completely frozen out

$$e^{-m/T} = 10^{-10^{16}} = 0$$



10^{-123}, $10^{-10^{16}}$

another great challenge?

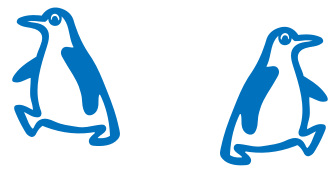
main hierarchy puzzle

$$m_{\text{quarks}}\,,\ m_{\text{leptons}} <<< \ E_{\text{Planck}}$$

its emergent physics solution:

$$m_{\text{quarks}} = m_{\text{leptons}} = 0$$

reason:
momentum-space **topology**
of quantum vacuum



cosmological constant puzzle

$$\Lambda <<<< \ E^4_{\text{Planck}}$$

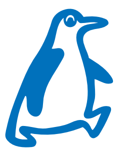
its emergent physics solution:

$$\Lambda = 0$$

reason:
thermodynamics
of quantum vacuum

Why no freezing at low T?

massless particles & gapless excitations are not frozen out

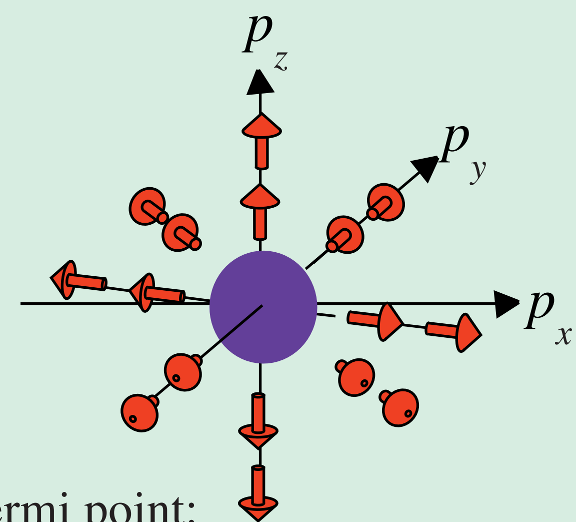


who protects massless excitations?

gapless fermions live
near
Fermi surface & Fermi point

who protects Fermi surface & Fermi point ?

Topology



Fermi point:
hedgehog in momentum space

we live because
Fermi point is the hedgehog
protected by topology

hedgehog is stable:
one cannot comb the hair
on a ball smooth

tools

Thermodynamics

responsible for properties
of vacuum energy

problems of cosmological constant:
perfect equilibrium Lorentz invariant vacuum
has $\Lambda = 0$;
perturbed vacuum has nonzero Λ
on order of perturbation

$$\Lambda <<<< E_{\text{Planck}}^4$$

Topology in momentum space

responsible for properties of
fermionic and bosonic quantum fields
in the background of quantum vacuum

Fermi point in momentum space
protected by topology is a source of
massless Weyl fermions, gauge fields & gravity

$$m_{\text{quarks}} , m_{\text{leptons}} <<< E_{\text{Planck}}$$

Quantum vacuum as topological substance: universality classes

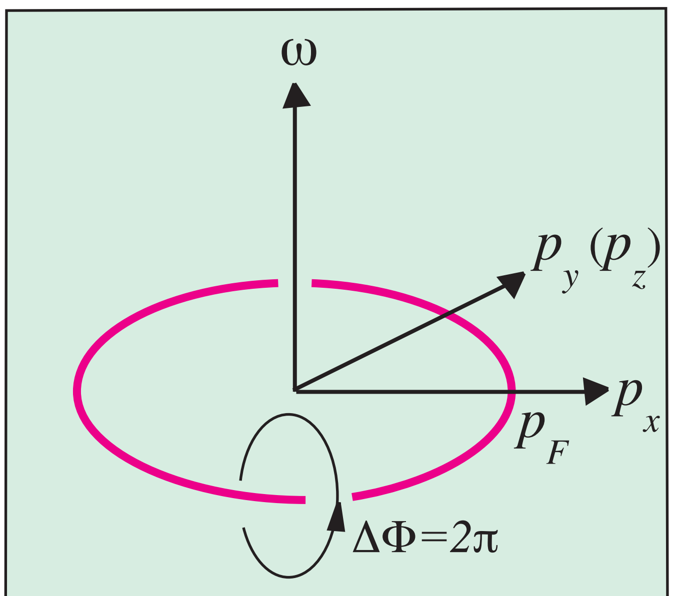
physics at low T is determined by gapless excitations

in metals low-lying excitations live near Fermi surface

or near Fermi point

universality classes of gapless vacua

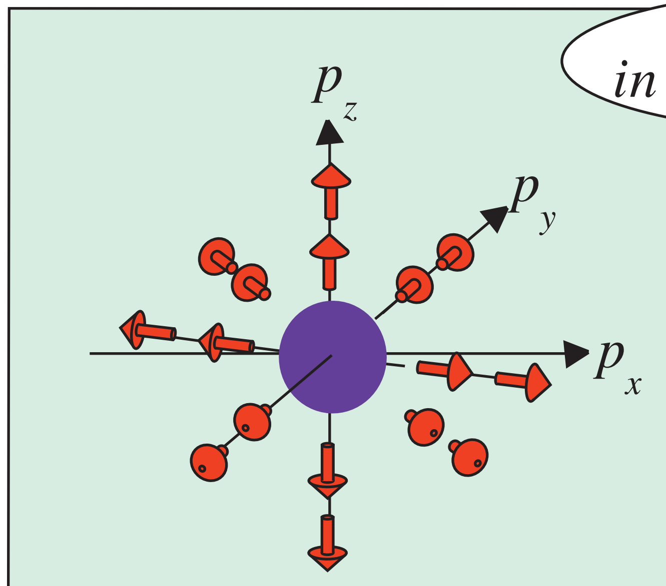
Fermi surface class



Fermi surface:
vortex line in **p**-space

Fermi point class

topology in momentum space



Fermi point:
hedgehog in **p**-space

topology is robust to deformations: nodes in spectrum survive interaction

Horava, Kitaev, Ludwig, Schnyder, Ryu, Furusaki, ...

2. Liquid ^{3}He & effective theory of vacuum with Fermi surface

two major universality classes of gapless fermionic vacua

Landau theory of Fermi liquid	Standard Model + gravity
vacuum with Fermi surface	**vacuum with Fermi point**

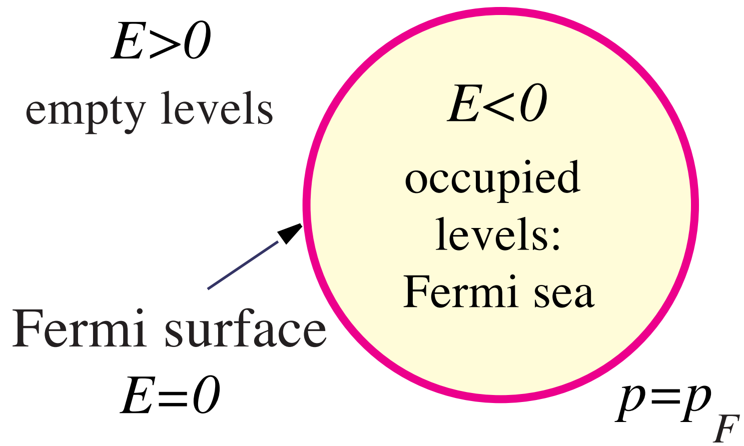
gravity emerges from
Fermi point
analog of
Fermi surface

→ $$g^{\mu\nu}(p_\mu - eA_\mu - e\tau\cdot\mathbf{W}_\mu)(p_\nu - eA_\nu - e\tau\cdot\mathbf{W}_\nu) = 0$$

Topological stability of Fermi surface

Energy spectrum of
non-interacting gas of fermionic atoms

$$E(p) = \frac{p^2}{2m} - \mu = \frac{p^2}{2m} - \frac{p_F^{\,2}}{2m}$$

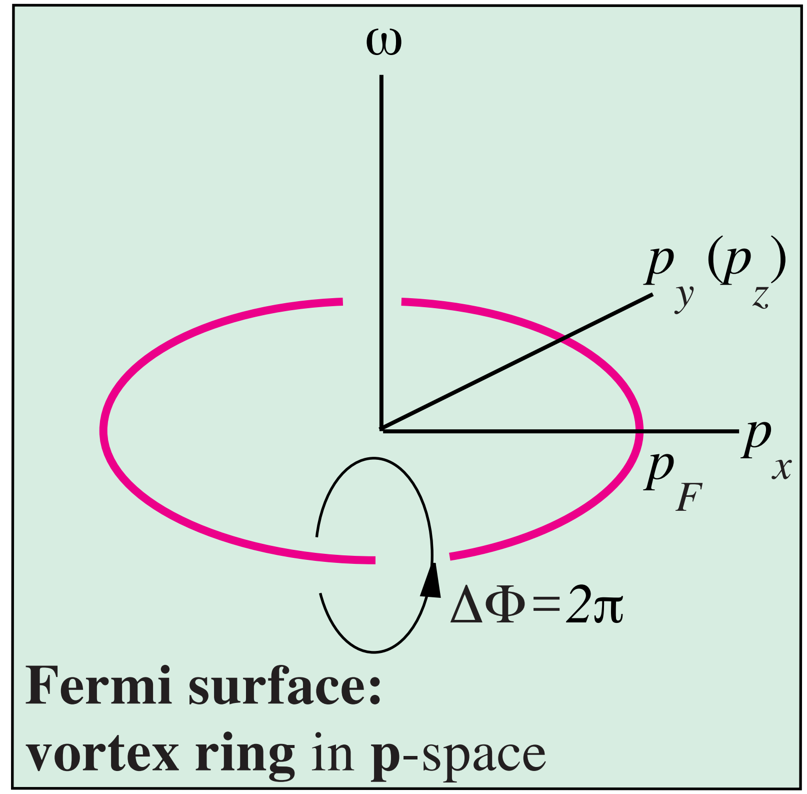


is Fermi surface a domain wall in momentum space?

no! it is a **vortex ring**

Green's function

$$G^{-1} = i\omega - E(p)$$



phase of Green's function

$$G(\omega,\mathbf{p}) = |G| e^{i\Phi}$$

has winding number $N = 1$

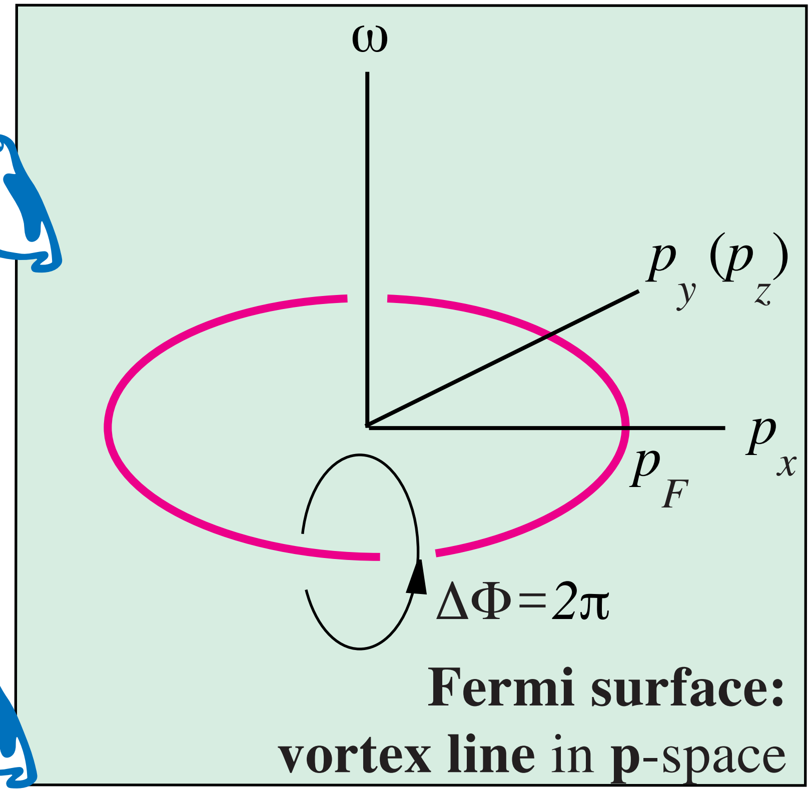
Route to Landau Fermi-liquid

is Fermi surface robust to interaction ?

Sure! Because of topology:
winding number N=1 *cannot change continuously,*
interaction cannot destroy singularity

then Fermi surface survives in Fermi liquid ?

Landau theory of Fermi liquid
is topologically protected & thus is universal



$G(\omega,\mathbf{p})=|G|e^{\,i\Phi}$

all metals have Fermi surface ...

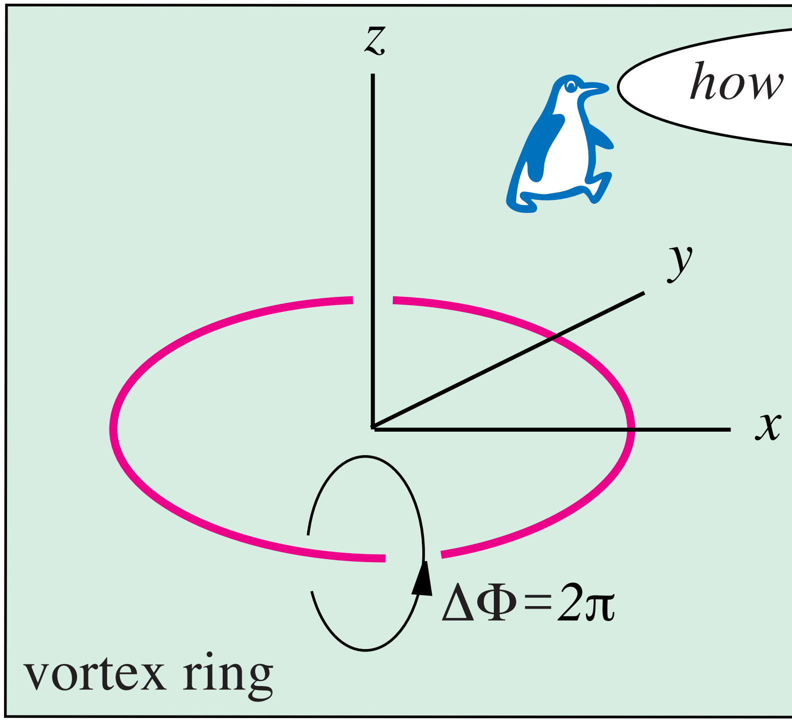
Not only metals.
Some superconductore too!

Stability conditions & Fermi surface topologies in a superconductor
Gubankova-Schmitt-Wilczek, *Phys.Rev*. **B**74 (2006) 064505

quantized vortex in **r**-space ≡ **Fermi surface** in **p**-space

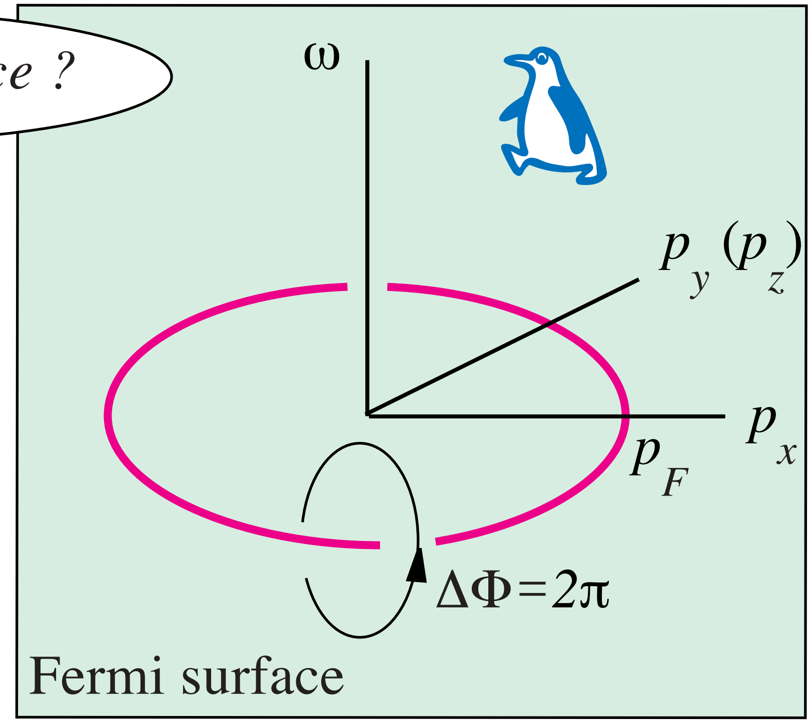
homotopy group π_1

Topology in **r**-space



how is it in ***p****-space ?*

Topology in **p**-space



winding number $N_1 = 1$

$\Psi(\mathbf{r}) = |\Psi|\, e^{i\Phi}$

scalar order parameter
of superfluid & superconductor

$G(\omega,\mathbf{p}) = |G| e^{i\Phi}$

Green's function (propagator)

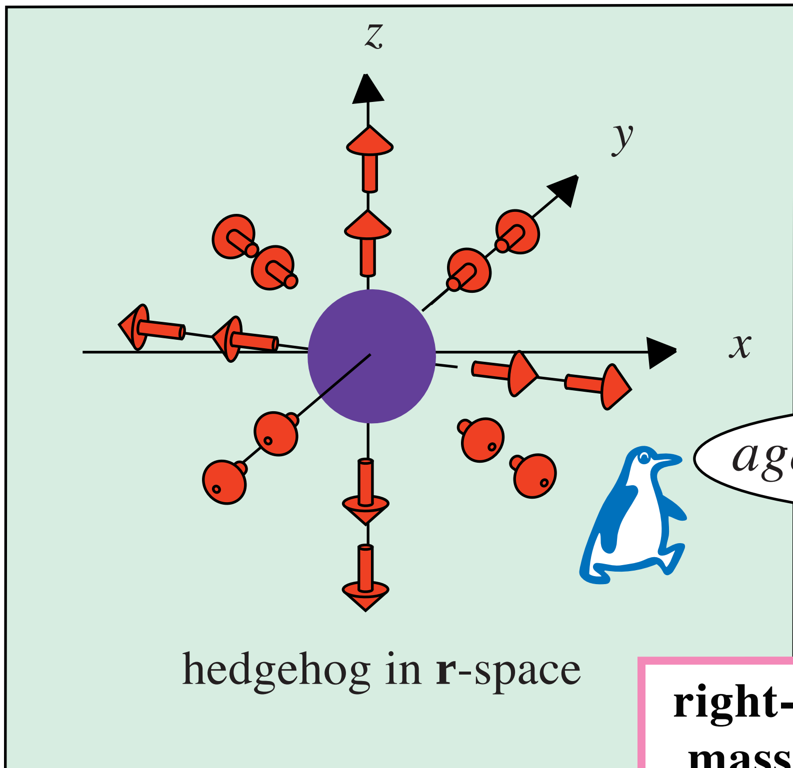
classes of mapping $S^1 \rightarrow U(1)$
manifold of
broken symmetry vacuum states

classes of mapping $S^1 \rightarrow GL(\mathrm{n,C})$
space of
non-degenerate complex matrices

3. Superfluid ^{3}He-A & Standard Model

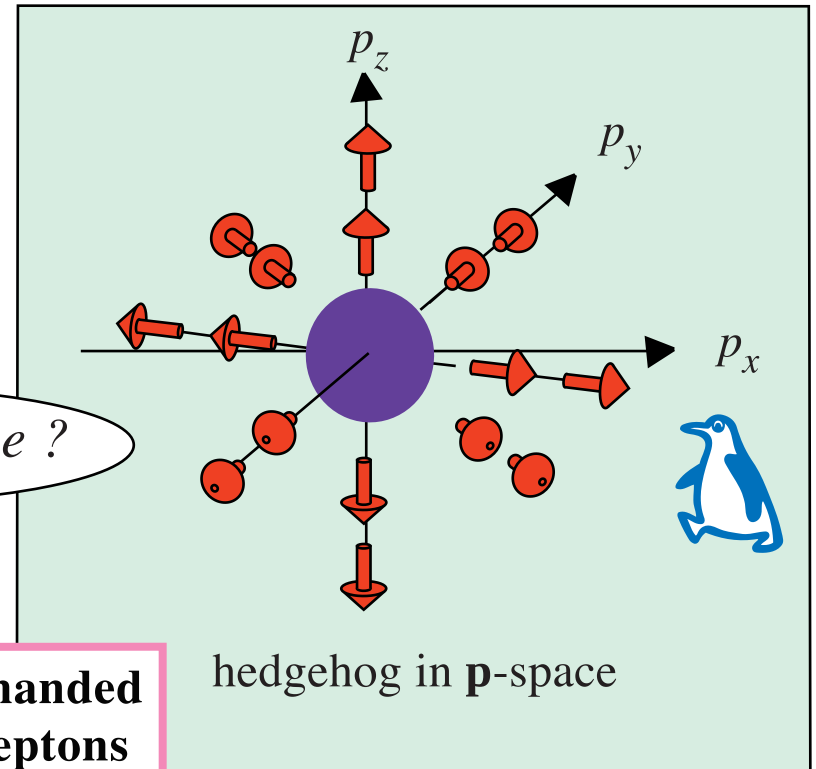
From Fermi surface to Fermi point

magnetic hedgehog vs right-handed electron



$$\sigma(\mathbf{r}) = \hat{\mathbf{r}}$$

again no difference ?



$$\sigma(\mathbf{p}) = \hat{\mathbf{p}}$$

right-handed and left-handed massless quarks and leptons are elementary particles in Standard Model

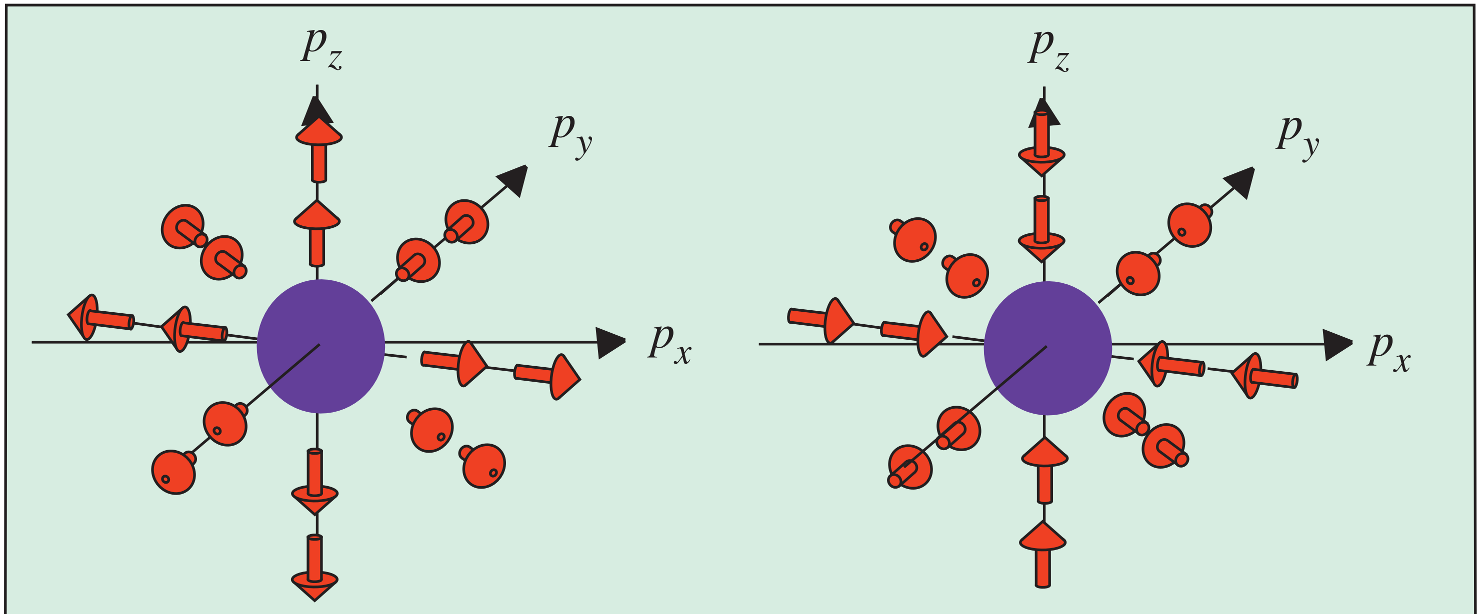
Landau CP symmetry is emergent

close to Fermi point

$$H = + c\, \sigma \cdot \mathbf{p}$$

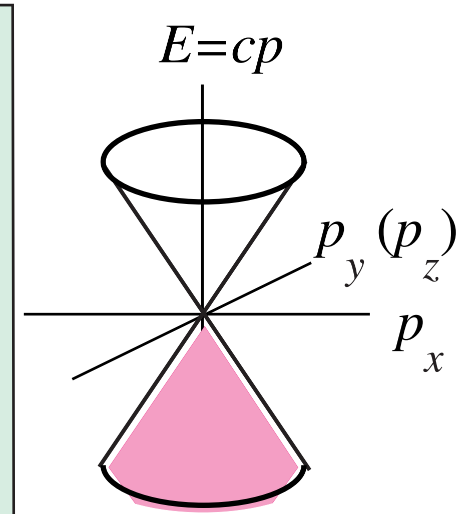
right-handed electron = hedgehog in **p**-space with spines = spins

Topological invariant for right-handed and left-handed elementary particles



hedgehog with spines (spins) outward ($N_3 = +1$)

hedgehog with spines (spins) inward ($N_3 = -1$)



right neutrino

$$H = +c\,\sigma \cdot \mathbf{p}$$
$$\mathbf{g}(\mathbf{p}) = +c\mathbf{p}$$

$$H = \sigma \cdot \mathbf{g}(\mathbf{p})$$

$$H = -c\,\sigma \cdot \mathbf{p}$$
$$\mathbf{g}(\mathbf{p}) = -c\mathbf{p}$$

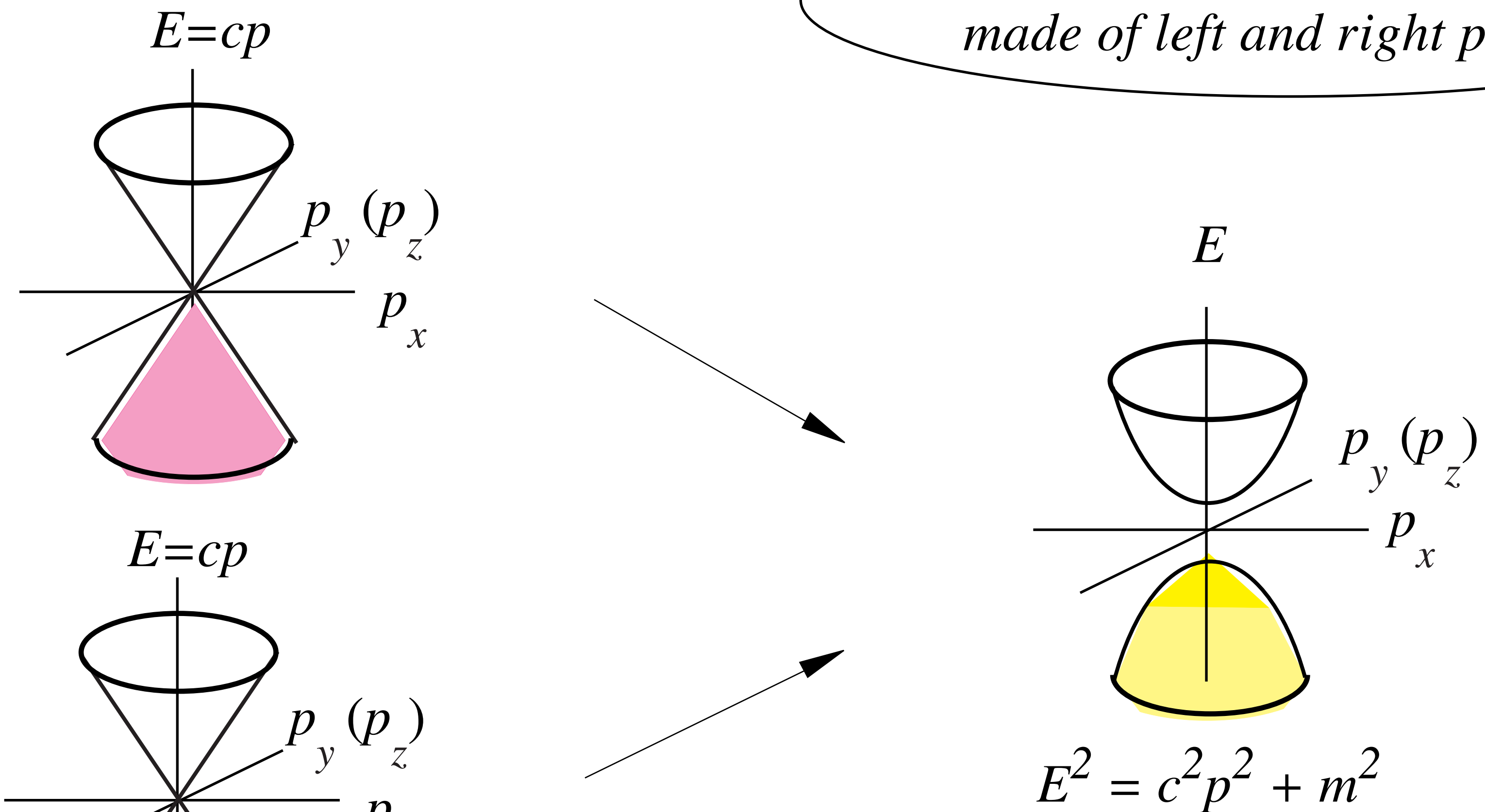
left neutrino

$$N_3 = \frac{1}{8\pi}\, e_{ijk} \int dS^i\, \hat{\mathbf{g}} \cdot (\partial^j \hat{\mathbf{g}} \times \partial^k \hat{\mathbf{g}})$$

over 2D surface around Fermi point

where are Dirac particles?

Dirac particle - composite object made of left and right particles



$$E^2 = c^2p^2 + m^2$$

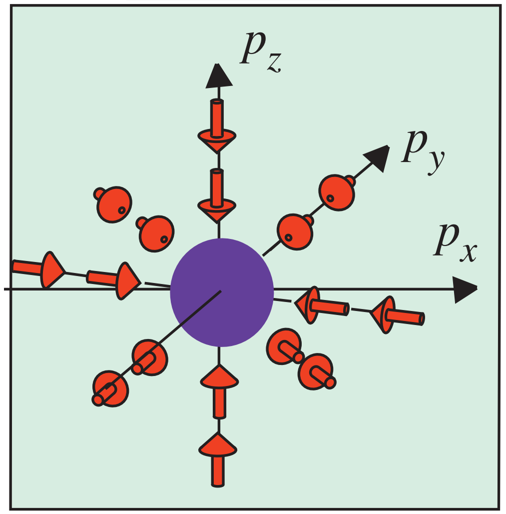
mixing of left and right particles is secondary effect, which occurs at extremely low temperature

$T_{ew} \sim 1$ TeV$\sim 10^{16}$K

Chiral fermions in Standard Model

Family #1 of quarks and leptons

left particles



hedgehog with spines (spins) inward $(N_3 = -1)$

quarks — $SU(3)_C$ — $SU(2)_L$

+2/3 u_L +1/6	-1/3 d_L +1/6
+2/3 u_L +1/6	-1/3 d_L +1/6
+2/3 u_L +1/6	-1/3 d_L +1/6

+2/3 u_R +2/3	-1/3 d_R -1/3
+2/3 u_R +2/3	-1/3 d_R -1/3
+2/3 u_R +2/3	-1/3 d_R -1/3

leptons

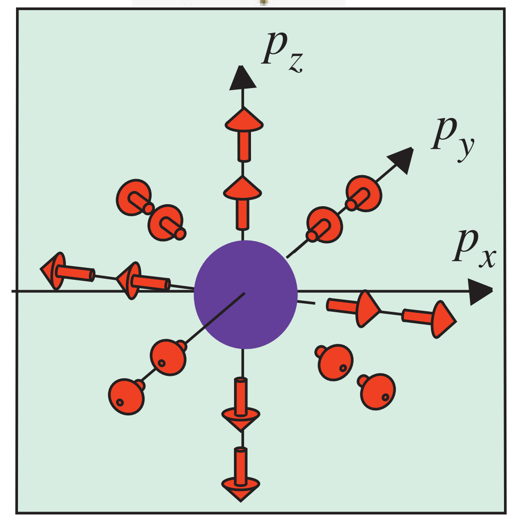
0 ν_L -1/2	-1 e_L -1/2

0 ν_R 0	-1 e_R -1

$$H = -\,c\,\sigma \cdot \mathbf{p}$$
$$N_3 = -1$$

$$H = +\,c\,\sigma \cdot \mathbf{p}$$
$$N_3 = +1$$

right particles

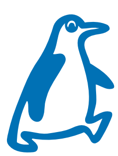


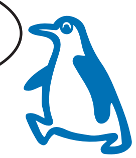
hedgehog with spines (spins) outward $(N_3 = +1)$

$$N_3 = \frac{1}{24\pi^2}\, e_{\mu\nu\lambda\gamma}\, \mathbf{tr} \int dS^\gamma\, \mathbf{G}^\mu\, \mathbf{G}^{-1}\, \mathbf{G}^\nu\, \mathbf{G}^{-1} \mathbf{G}^\lambda\, \mathbf{G}^{-1}$$

over 3D surface S in 4D momentum space

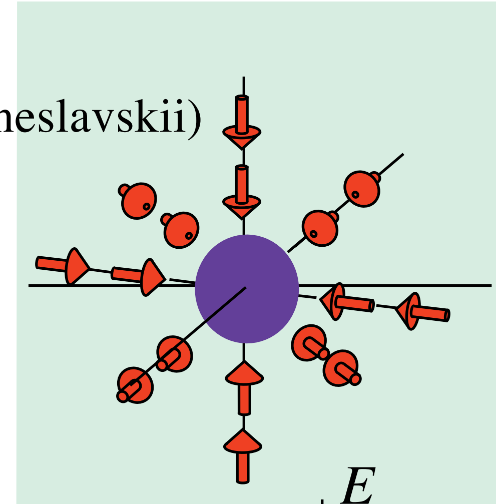
general topological invariant in terms of Green's function





Fermi (Dirac) points in 3+1 gapless topological matter

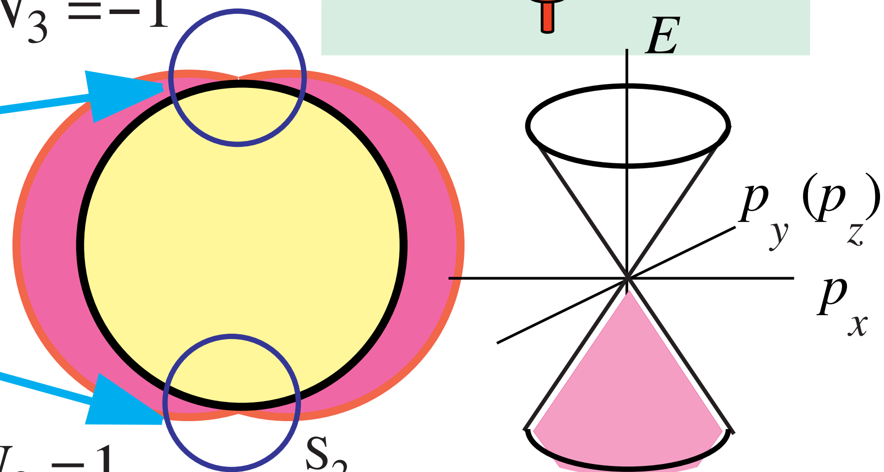
topologically protected point nodes in:
superfluid ^{3}He-A, triplet cold Fermi gases, semi-metal (Abrikosov-Beneslavskii)



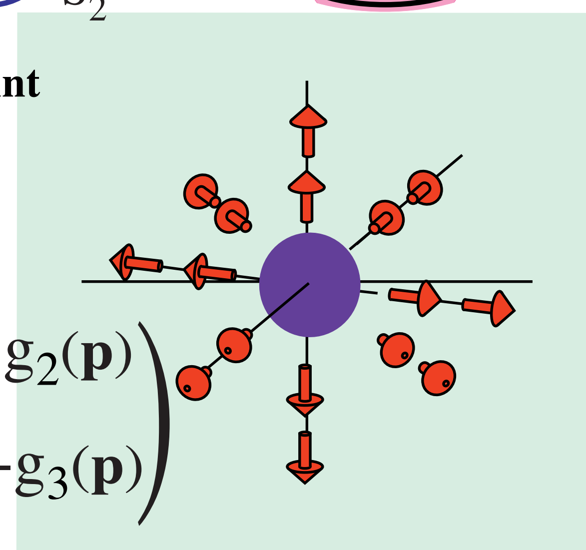
Gap node - Fermi point
(anti-hedgehog)

$$N_3 = \frac{1}{8\pi} e_{ijk} \int dS^k \; \hat{\mathbf{g}} \cdot (\partial_{p_i} \hat{\mathbf{g}} \times \partial_{p_j} \hat{\mathbf{g}})$$

over 2D surface S
in 3D p-space



Gap node - Fermi point
(hedgehog)



$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix}$$

emergence of relativistic QFT near Fermi (Dirac) point

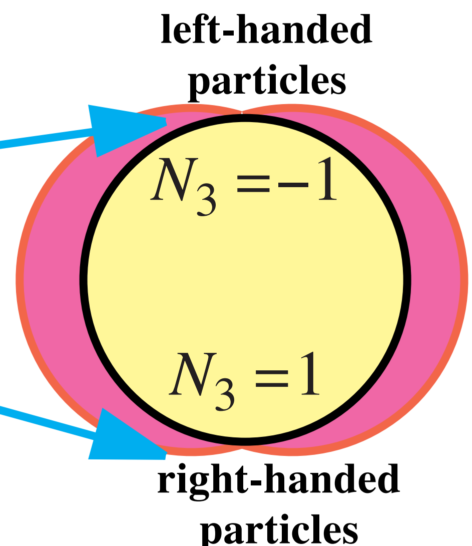
original non-relativistic Hamiltonian

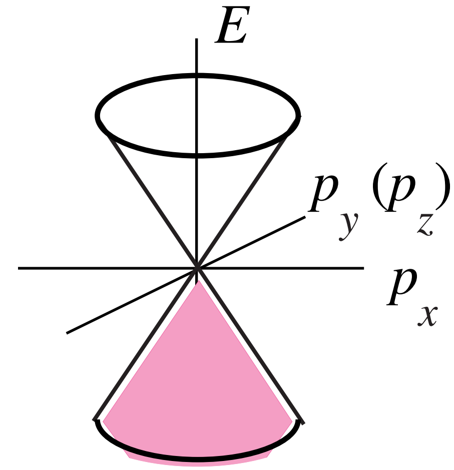
$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix} = \boldsymbol{\tau} \cdot \mathbf{g}(\mathbf{p})$$

close to nodes, i.e. in low-energy corner
relativistic chiral fermions emerge

$$H = N_3\, c\, \boldsymbol{\tau} \cdot \mathbf{p}$$

$$E = \pm\, cp$$





chirality is emergent ??

what else is emergent ?

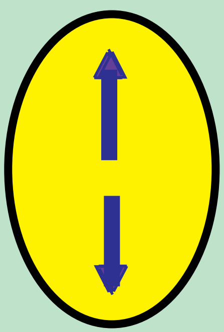
top. invariant determines chirality in low-energy corner

relativistic invariance as well

bosonic collective modes in two generic fermionic vacua

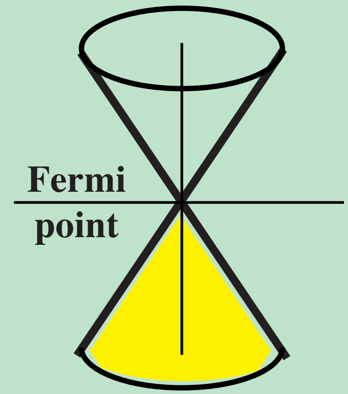
Landau theory of Fermi liquid

collective Bose modes
of fermionic vacuum:
propagating
oscillation of **shape**
of Fermi surface



Landau, ZhETF **32**, 59 (1957)

Standard Model + gravity

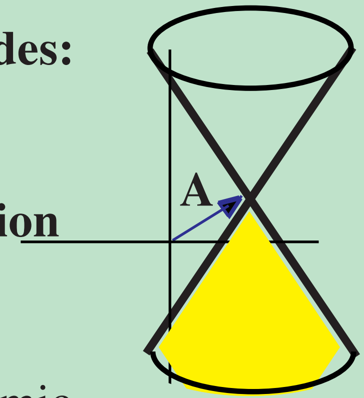


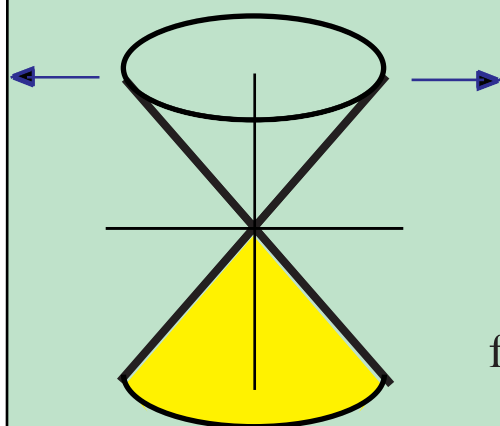
collective Bose modes:

propagating
oscillation of **position**
of Fermi point

$$\mathbf{p} \rightarrow \mathbf{p} - e\mathbf{A}$$

form effective dynamic
electromagnetic field

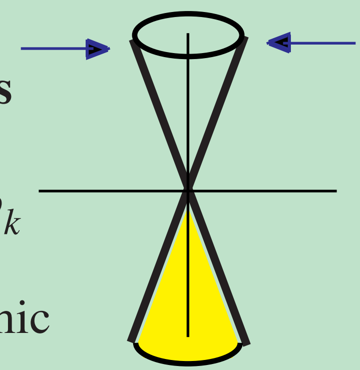




propagating
oscillation of **slopes**

$$E^2 = c^2 p^2 \rightarrow g^{ik} p_i p_k$$

form effective dynamic
gravity field



two generic quantum field theories of interacting bosonic & fermionic fields

relativistic quantum fields and gravity emerging near Fermi point

Atiyah-Bott-Shapiro construction:
linear expansion of Hamiltonian near the nodes in terms of Dirac Γ-matrices

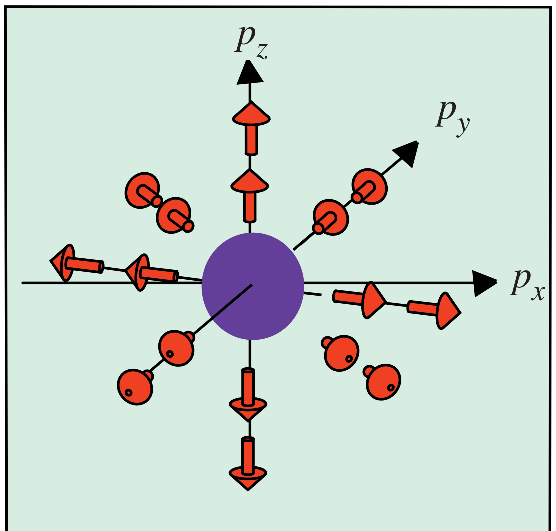
$$E = v_F\,(p - p_F)$$

linear expansion near Fermi surface

emergent relativity

$$H = e_i^k\,\Gamma^i \cdot (p_k - p_k^0)$$

linear expansion near Fermi point



hedgehog in **p**-space

$$g^{\mu\nu}(p_\mu - eA_\mu - e\tau\cdot\mathbf{W}_\mu)(p_\nu - eA_\nu - e\tau\cdot\mathbf{W}_\nu) = 0$$

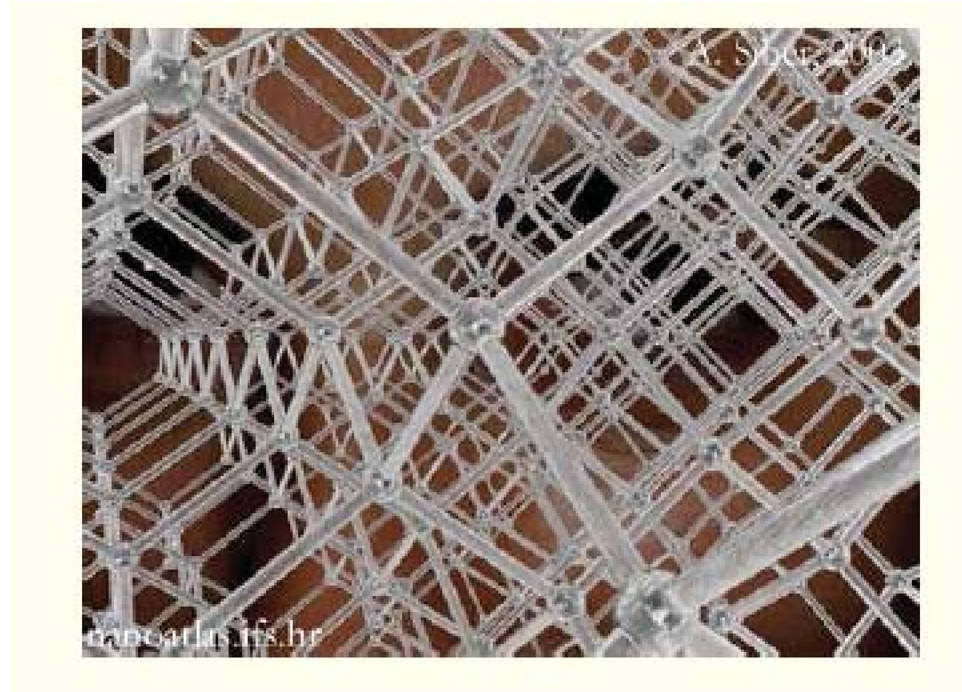
- $g^{\mu\nu}$: effective metric: emergent gravity
- A_μ: effective electromagnetic field
- $\mathbf{W}_\mu$: effective $SU(2)$ gauge field
- e: effective electric charge $e = +1$ or -1
- τ: effective isotopic spin

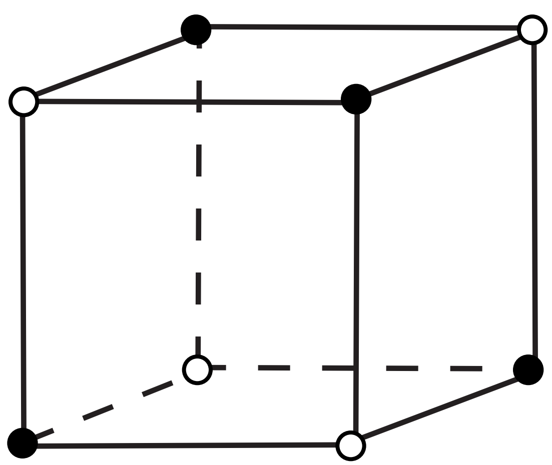
all ingredients of Standard Model :
chiral fermions & gauge fields
emerge in low-energy corner
together with spin, Dirac Γ−matrices, gravity & physical laws:
Lorentz & gauge invariance, equivalence principle, etc

gravity & gauge fields are collective modes of vacua with Fermi point

quantum vacuum as cryo-crystal

4D graphene
Michael Creutz JHEP 04 (2008) 017





● Fermi (Dirac) points with $N_3 = +1$

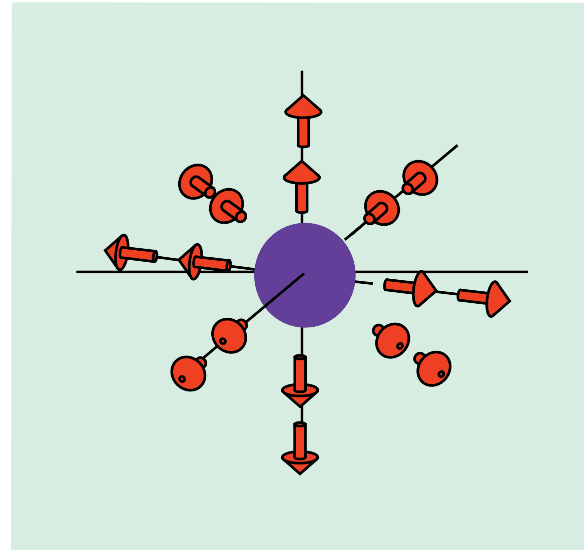
○ Fermi (Dirac) points with $N_3 = -1$

4. From Fermi point to intrinsic QHE & topological insulators

$$N_3 = \frac{1}{8\pi} e_{ijk} \int dS^k \ \hat{\mathbf{g}} \cdot (\partial_{p_i} \hat{\mathbf{g}} \times \partial_{p_j} \hat{\mathbf{g}})$$

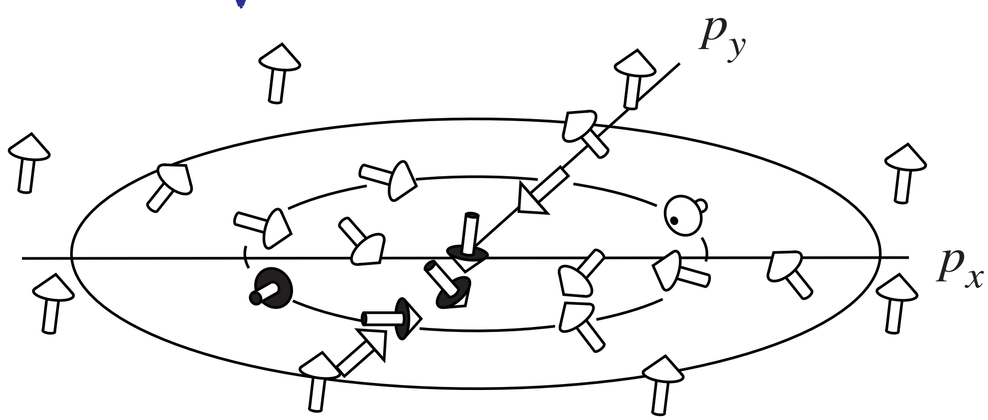
over 2D surface S
in 3D momentum space

3+1vacuum with Fermi point



dimensional reduction

Fully gapped 2+1 system



$$\tilde{N}_3 = \frac{1}{4\pi} \int dp_x dp_y \ \hat{\mathbf{g}} \cdot (\partial_{p_x} \hat{\mathbf{g}} \times \partial_{p_y} \hat{\mathbf{g}})$$

over the whole 2D momentum space
or over 2D Brillouin zone

topological insulators & superconductors in 2+1

p-wave 2D superconductor, ^{3}He-A film, HgTe insulator quantum well

$$\mathrm{H} = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

$$p^2 = p_x{}^2 + p_y{}^2$$

How to extract useful information on energy states from Hamiltonian without solving equation

$$\mathrm{H}\psi = E\psi$$

Topological invariant in momentum space

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

$$p^2 = p_x^{\,2} + p_y^{\,2}$$

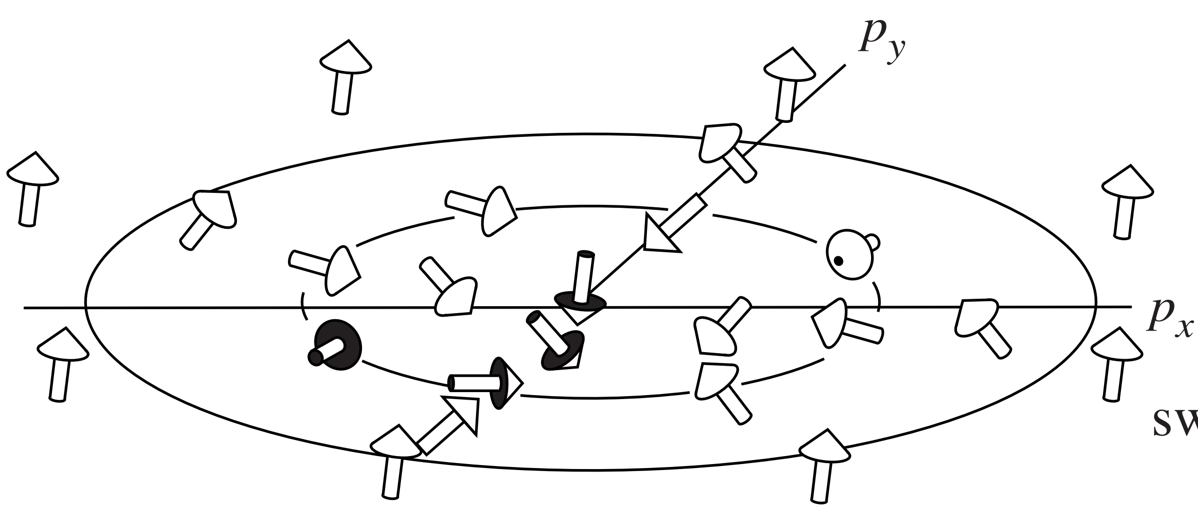
$$H = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix} = \boldsymbol{\tau} \cdot \mathbf{g}(\mathbf{p})$$

fully gapped 2D state at $\mu \neq 0$

$$\tilde{N}_3 = \frac{1}{4\pi} \int d^2p \; \hat{\mathbf{g}} \cdot (\partial_{p_x} \hat{\mathbf{g}} \times \partial_{p_y} \hat{\mathbf{g}})$$

GV, JETP **67**, 1804 (1988)

Skyrmion (coreless vortex) in momentum space at $\mu > 0$

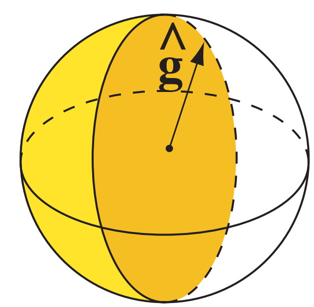


unit vector

$\hat{\mathbf{g}}(p_x, p_y)$

sweeps unit sphere

$\tilde{N}_3\ (\mu > 0) = 1$

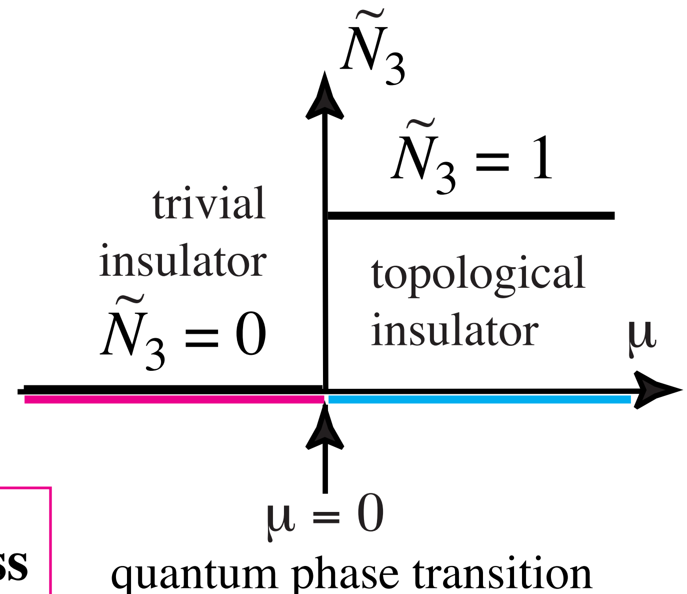


quantum phase transition:
from topological to non-topologicval insulator/superconductor

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix} = \boldsymbol{\tau} \cdot \mathbf{g}(\mathbf{p})$$

Topological invariant in momentum space

$$\widetilde{N}_3 = \frac{1}{4\pi} \int d^2p \; \hat{\mathbf{g}} \cdot (\boldsymbol{\partial}_{p_x} \hat{\mathbf{g}} \times \boldsymbol{\partial}_{p_y} \hat{\mathbf{g}})$$



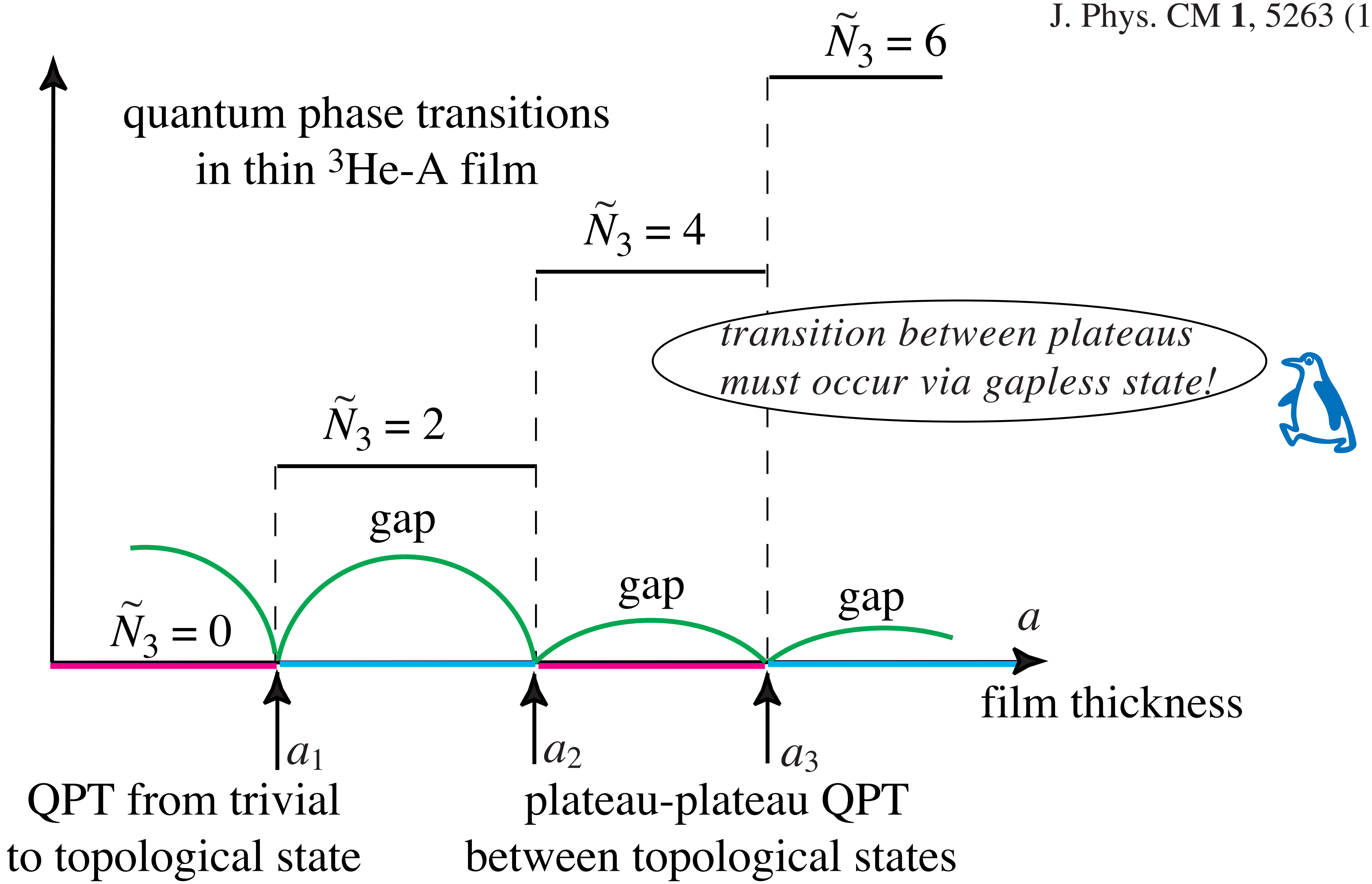
intermediate state at $\mu = 0$ must be gapless

$\Delta\widetilde{N}_3 \neq 0$ is origin of fermion zero modes
at the interface between states with different $\widetilde{N}_3$

p-space invariant in terms of Green's function & topological QPT

$$\tilde{N}_3 = \frac{1}{24\pi^2}\, e_{\mu\nu\lambda}\, \mathbf{tr} \int d^2p\, d\omega\, \mathbf{G}\, \partial^\mu \mathbf{G}^{-1}\, \mathbf{G}\, \partial^\nu \mathbf{G}^{-1} \mathbf{G}\, \partial^\lambda \mathbf{G}^{-1}$$

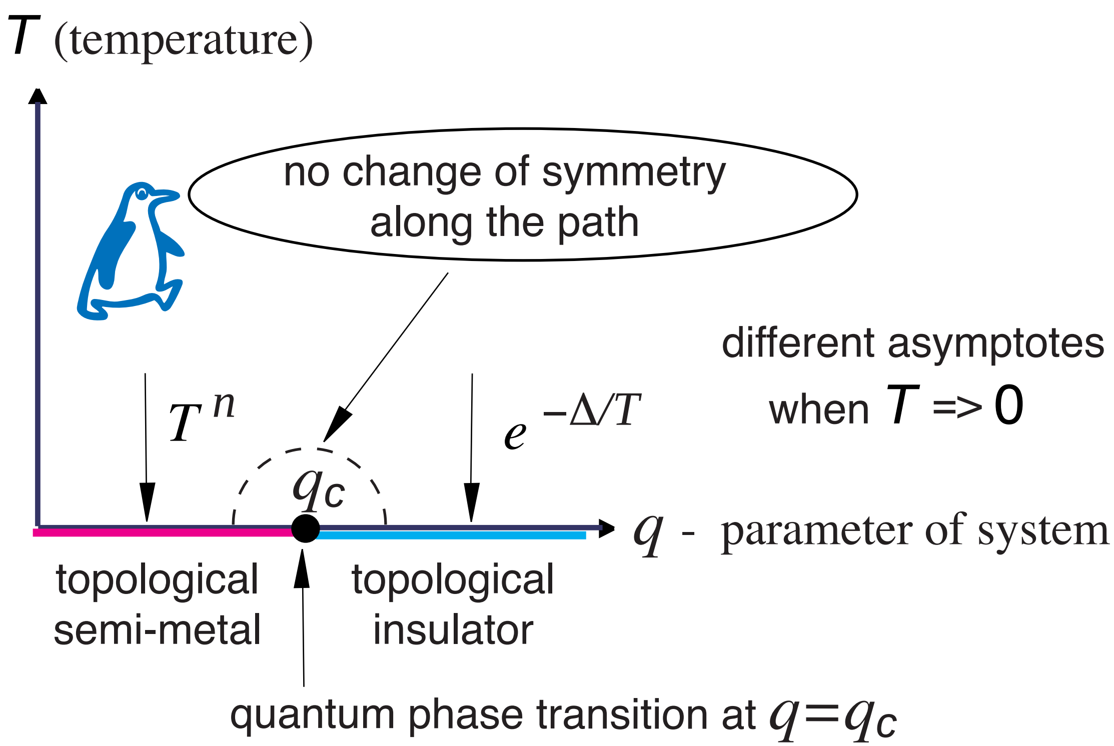
GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)



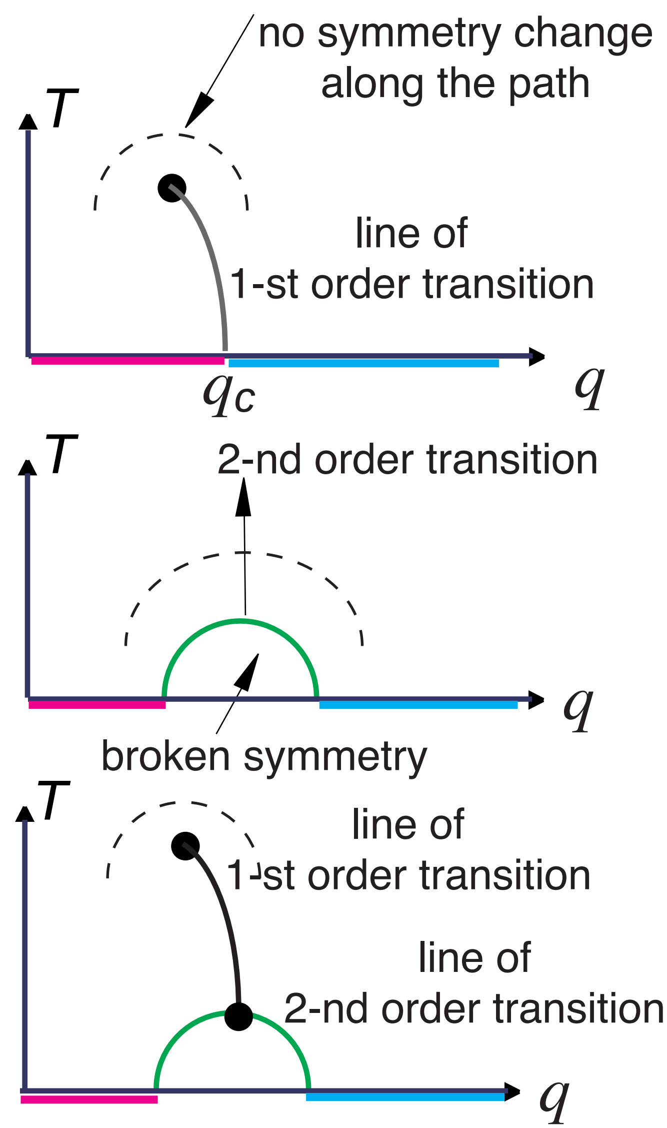
topological quantum phase transitions

transitions between **ground states (vacua)** of the **same symmetry**,
but **different topology** in **momentum space**

example: QPT between gapless & gapped matter



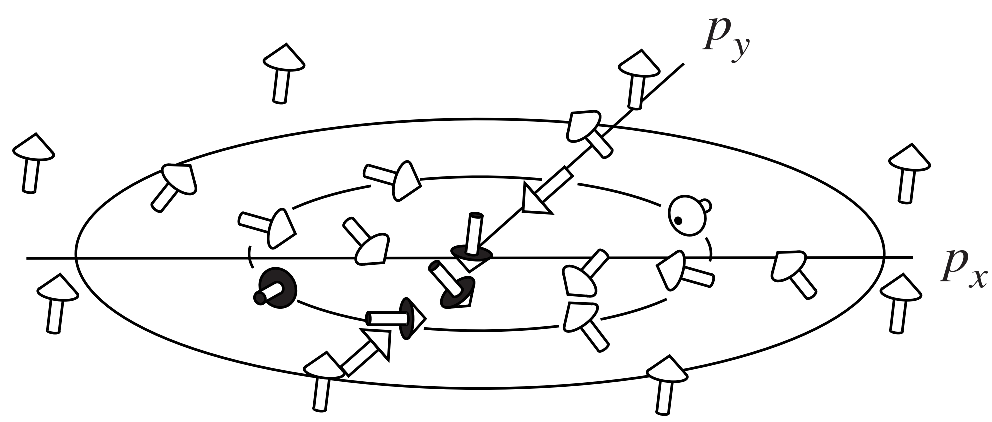
QPT interrupted
by thermodynamic transitions



other topological QPT:
Lifshitz transition,
transtion between topological and nontopological superfluids,
plateau transitions,
confinement-deconfinement transition, ...

Zero energy states on surface of topological insulators & superfluids

Fully gapped 2+1 system

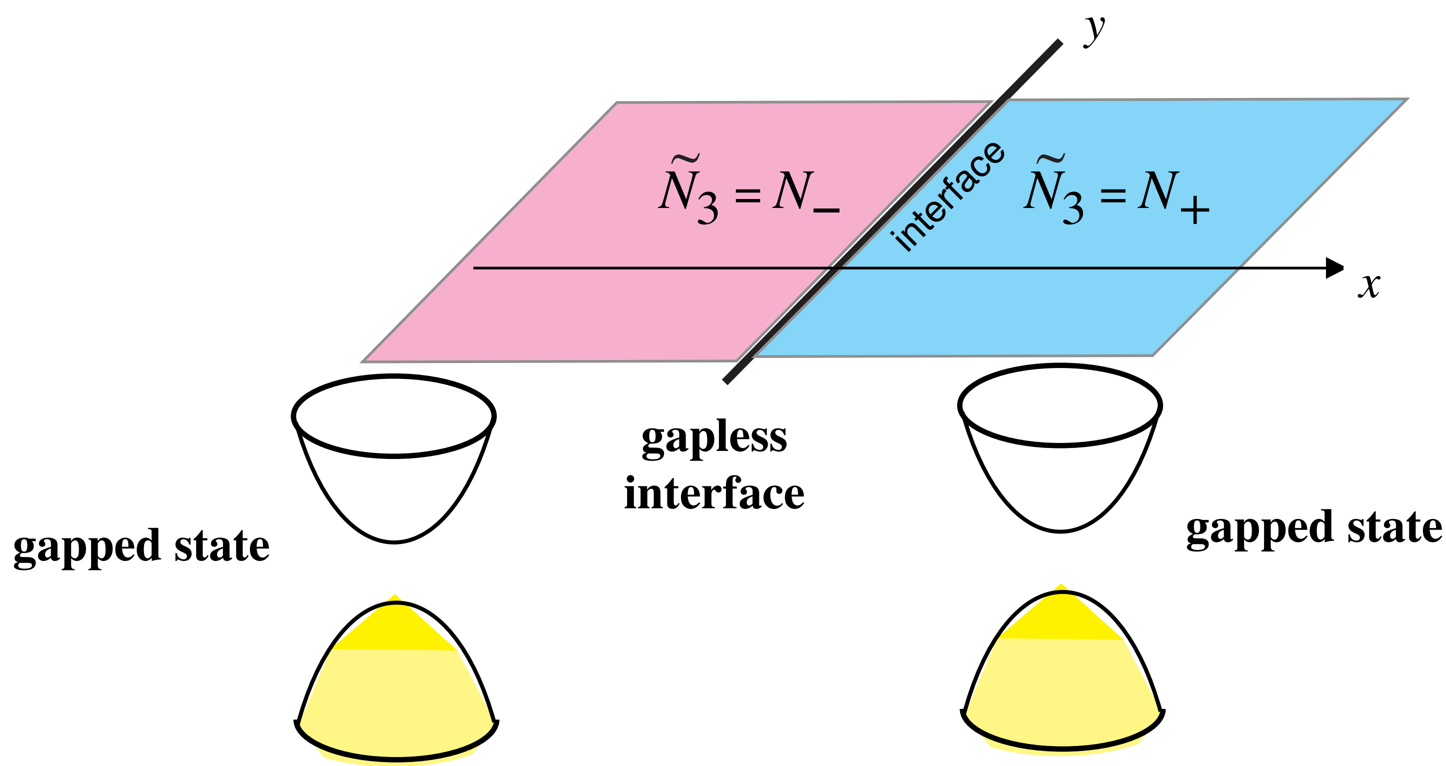


$$\tilde{N}_3 = \frac{1}{4\pi} \int dp_x dp_y \; \hat{\mathbf{g}} \cdot (\partial_{p_x} \hat{\mathbf{g}} \times \partial_{p_y} \hat{\mathbf{g}})$$

Fully gapped 3+1 system

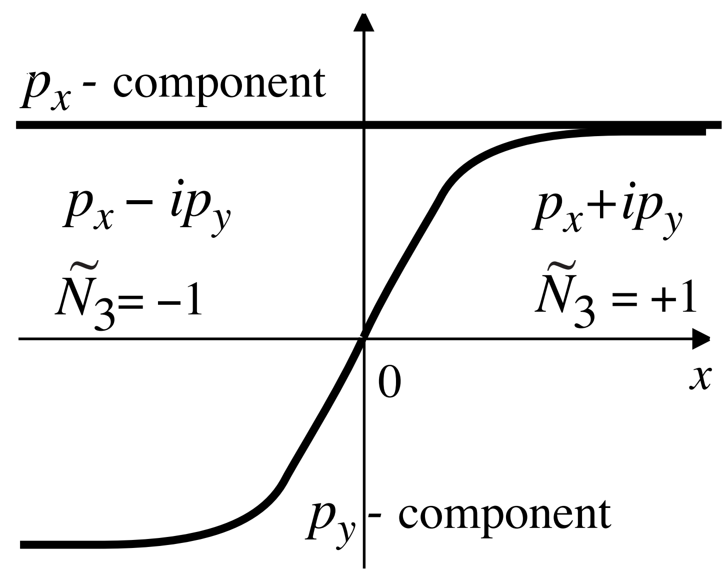
Majorana fermions on the surface and in the vortex cores

interface between two 2+1 topological insulators or gapped superfluids

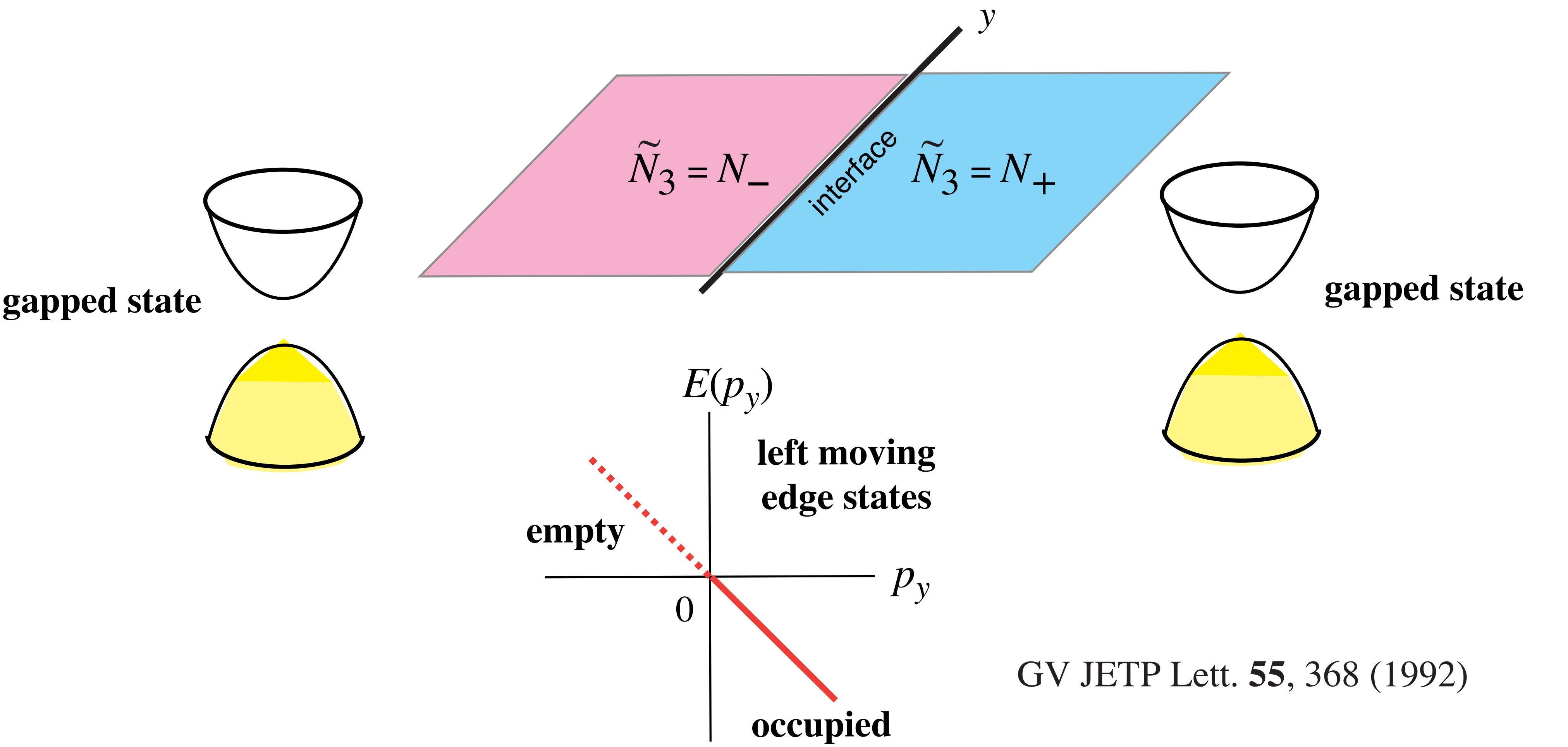


* domain wall in 2D chiral superconductors:

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + i\, p_y \tanh x) \\ c(p_x - i\, p_y \tanh x) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$



Edge states at interface between two 2+1 topological insulators or gapped superfluids

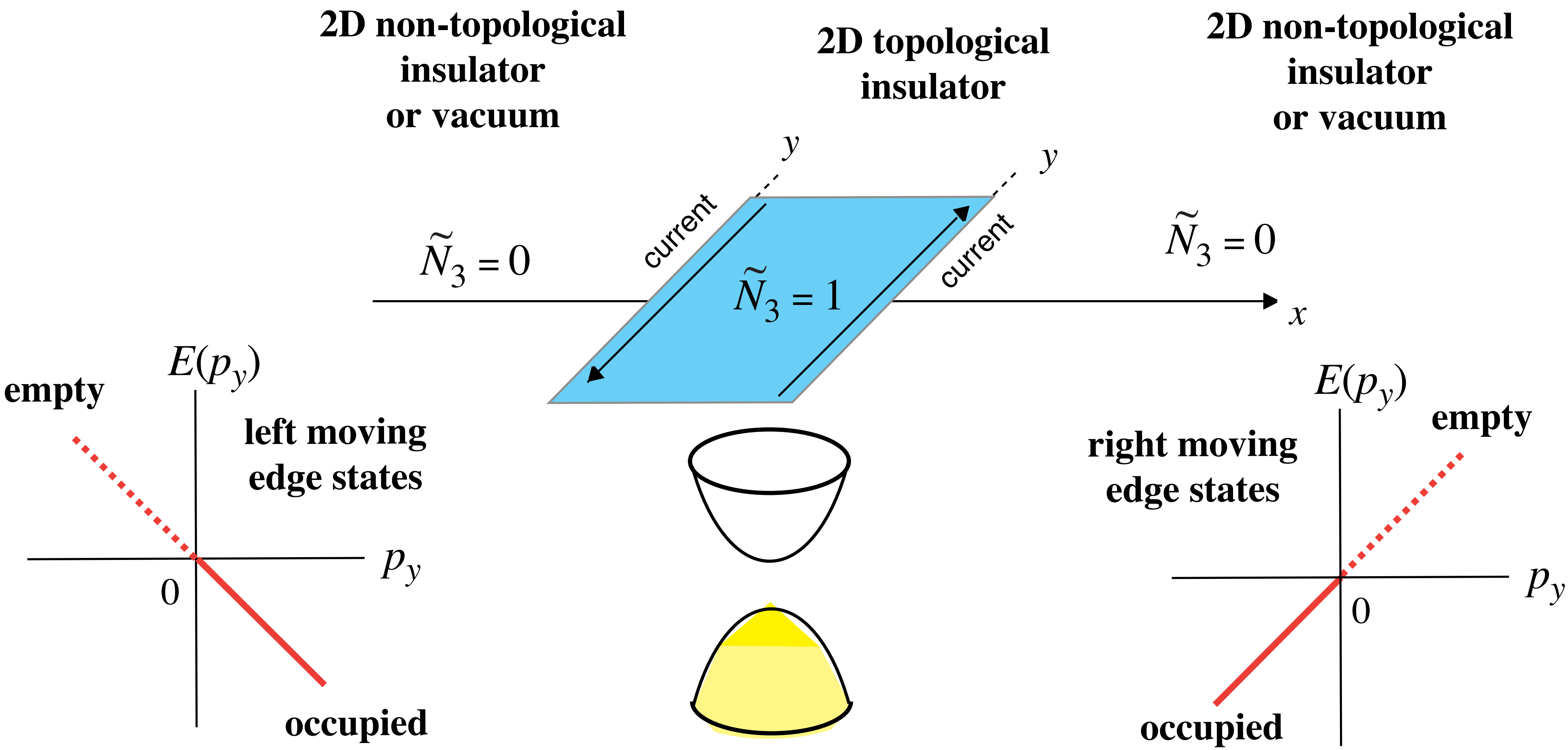


GV JETP Lett. **55**, 368 (1992)

Index theorem:
number of fermion zero modes
at interface:

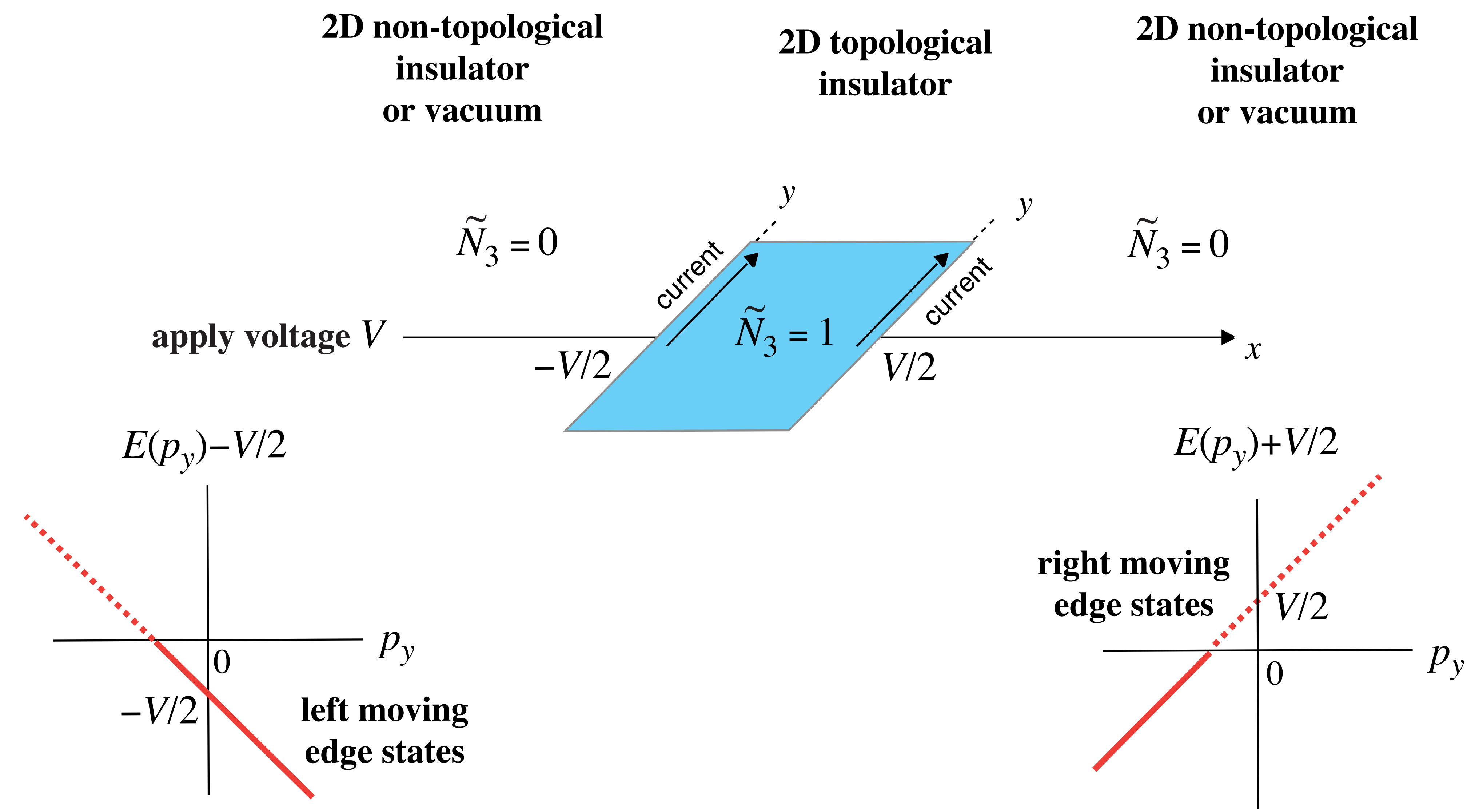
$$\nu = N_+ - N_-$$

Edge states and currents



current $J_y = J_{\text{left}} + J_{\text{right}} = 0$

Edge states and Quantum Hall effect



current $J_y = J_{\text{left}} + J_{\text{right}} = \sigma_{xy} E_y$

$$\sigma_{xy} = \frac{e^2}{4\pi} \tilde{N}_3$$

Intrinsic spin-current quantum Hall effect & momentum-space invariant

$$\text{spin current}\quad J_x^{\,z} = \frac{1}{4\pi}\left(\gamma N_{ss}\, dH^z/dy + N_{se}\, E_y\right)$$

spin-spin QHE (N_{ss})

spin-charge QHE (N_{se})

2D singlet superconductor:

$$\sigma_{xy}^{\text{spin/spin}} = \frac{N_{ss}}{4\pi}$$

s-wave: $N_{ss} = 0$

$p_x + ip_y$: $N_{ss} = 2$

$d_{xx\text{-}yy} + id_{xy}$: $N_{ss} = 4$

film of planar phase of superfluid ^{3}He

$$\sigma_{xy}^{\text{spin/charge}} = \frac{N_{se}}{4\pi}$$

GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)

planar phase film of 3He & 2D topological insulator

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + i\, p_y \sigma_z) \\ c(p_x - i\, p_y \sigma_z) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

$$\tilde{N}_3 = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \mathbf{tr} \left[\int d^2p\, d\omega\, \mathbf{G}\, \boldsymbol{\partial}^\mu\, \mathbf{G}^{-1}\, \mathbf{G}\, \boldsymbol{\partial}^\nu\, \mathbf{G}^{-1} \mathbf{G}\, \boldsymbol{\partial}^\lambda\, \mathbf{G}^{-1} \right] = 0$$

$$\tilde{N}_{se} = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \mathbf{tr} \left[\int d^2p\, d\omega\, \sigma_z\, \mathbf{G}\, \boldsymbol{\partial}^\mu\, \mathbf{G}^{-1}\, \mathbf{G}\, \boldsymbol{\partial}^\nu\, \mathbf{G}^{-1} \mathbf{G}\, \boldsymbol{\partial}^\lambda\, \mathbf{G}^{-1} \right]$$

$\tilde{N}_3^+ = +1 \quad \tilde{N}_3^- = -1$

$\tilde{N}_3 = \tilde{N}_3^+ + \tilde{N}_3^- = 0 \qquad \tilde{N}_{se} = \tilde{N}_3^+ - \tilde{N}_3^- = 2$

spin quantum Hall effect

spin current $J_x^z = \frac{1}{4\pi} N_{se} E_y$

spin-charge QHE

$$\sigma_{xy}^{\text{spin/charge}} = \frac{N_{se}}{4\pi}$$

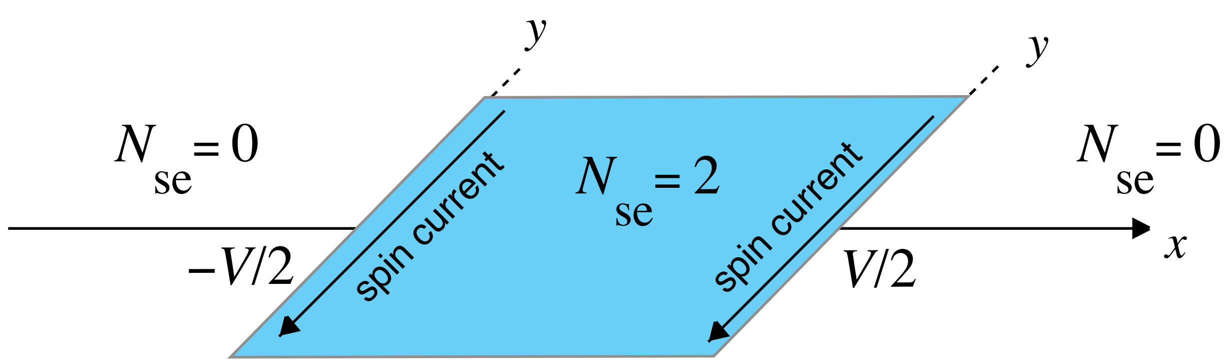
$N_{se} = 2$

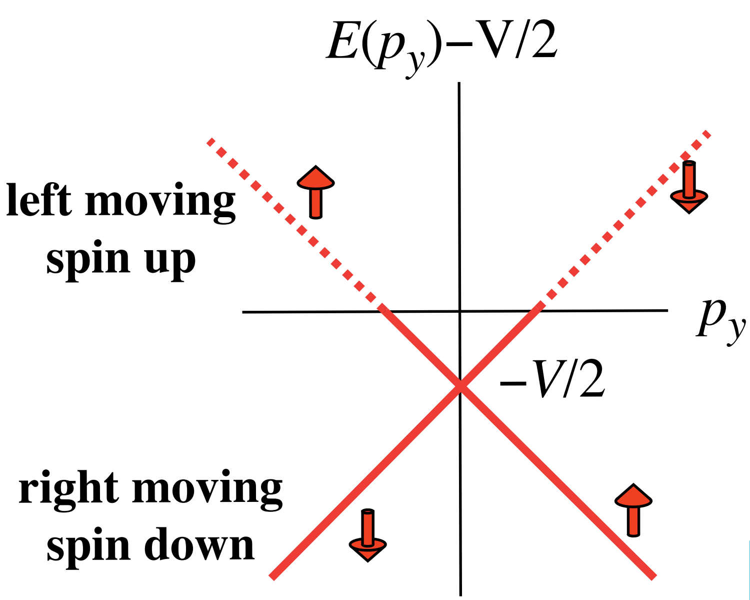
GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)

Intrinsic spin-current quantum Hall effect & edge state

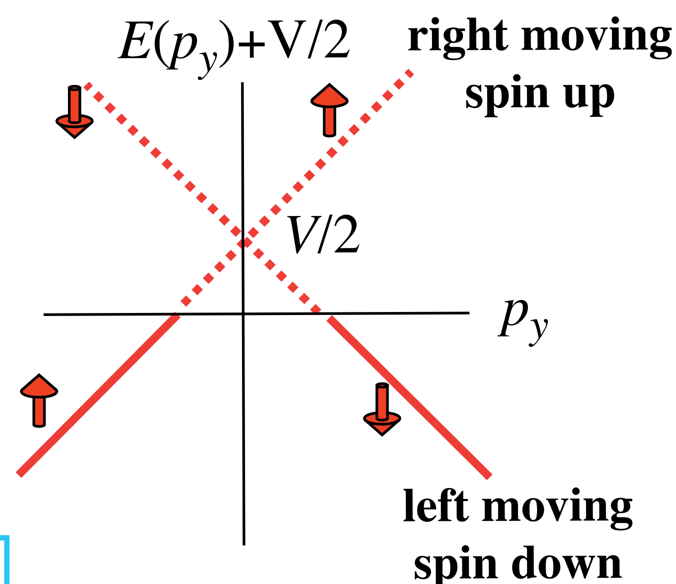
spin current $J_x^{\,z} = \dfrac{1}{4\pi}\left(\gamma N_{ss}\, dH^z/dy + N_{se} E_y\right)$

spin-charge QHE





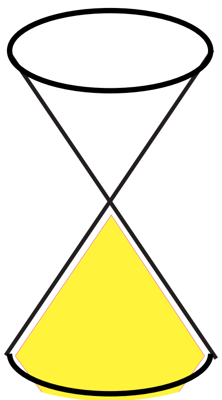
$$\sigma_{xy}^{\text{spin/charge}} = \frac{N_{se}}{4\pi}$$



electric current is zero
spin current is nonzero

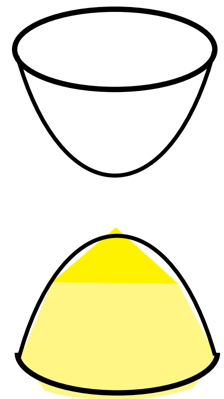
3D topological superfluids / insulators / semiconductors / vacua

gapless topologically
nontrivial vacua

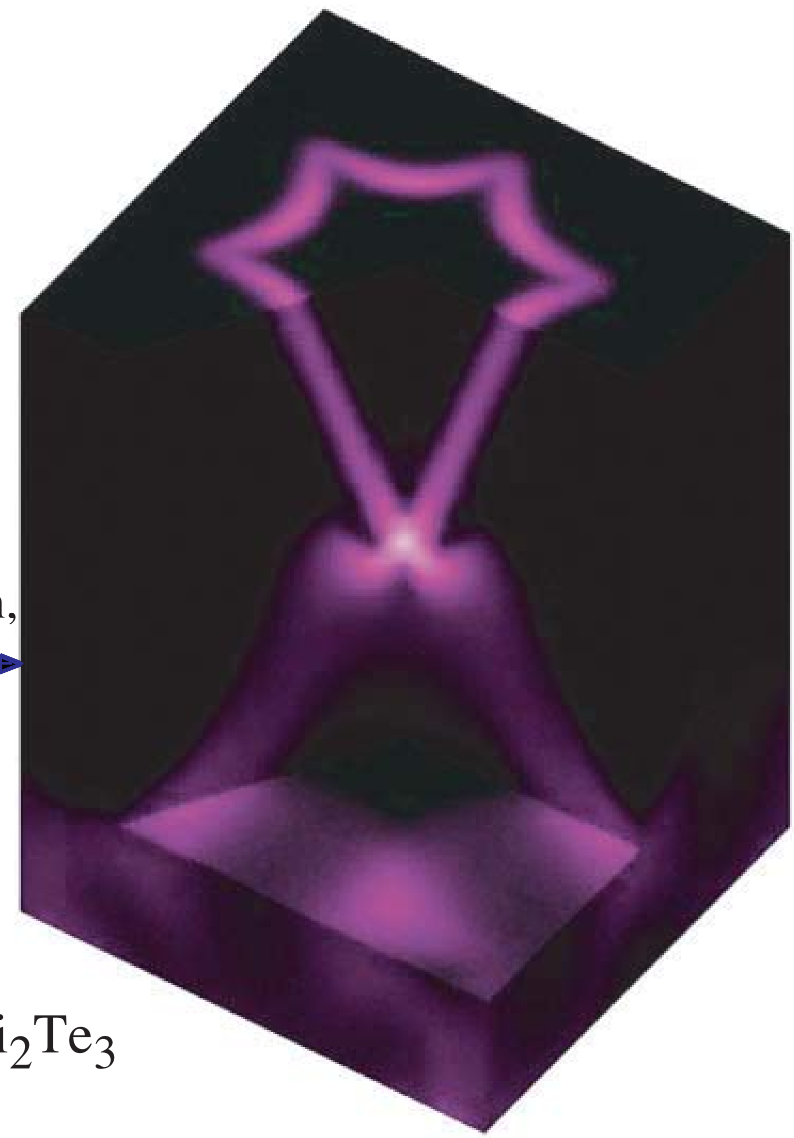


3He-A,
Standard Model
above electroweak transition,
semimetals,
4D graphene
(cryocrystalline vacuum)

fully gapped topologically
nontrivial vacua

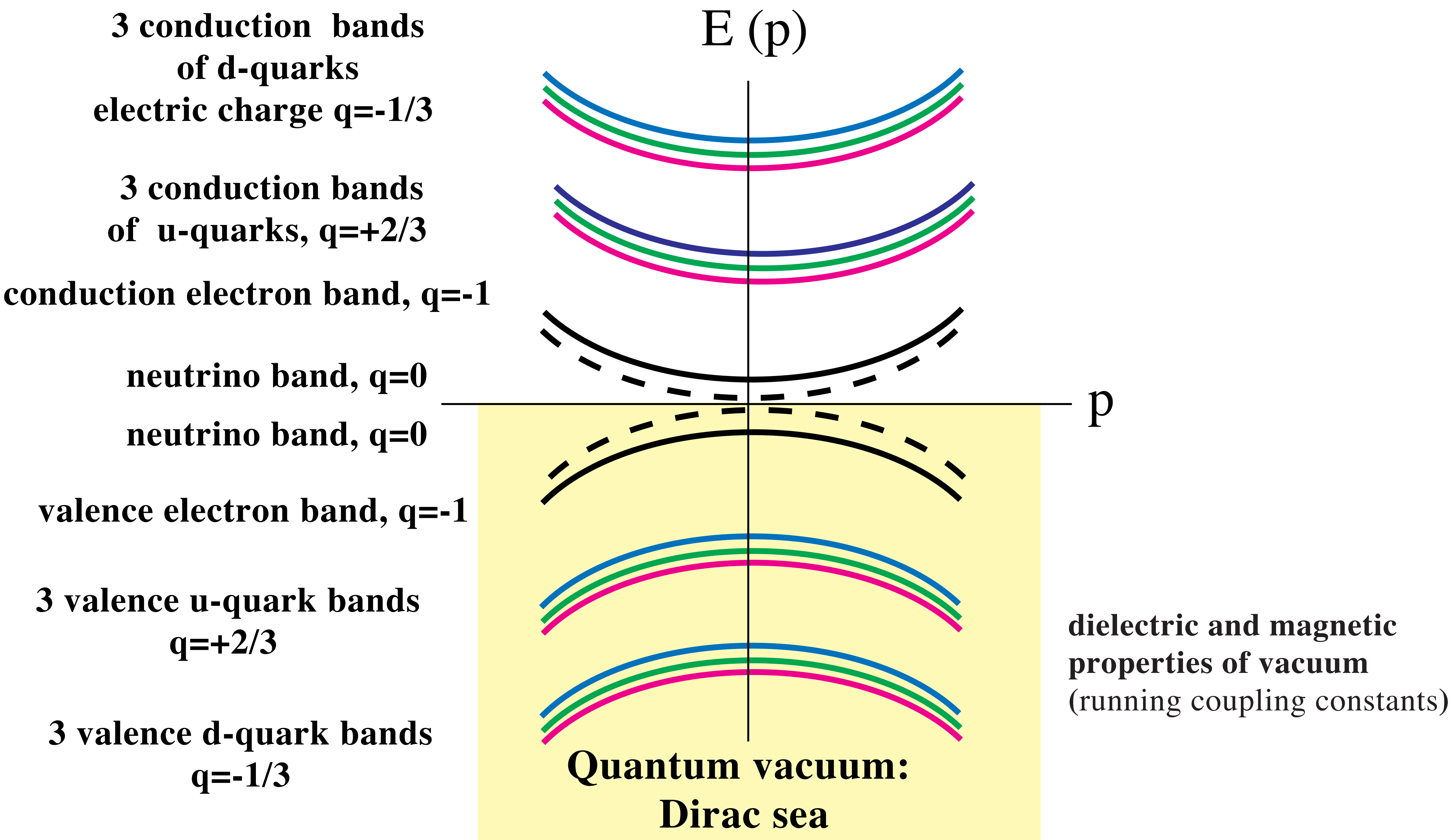


3He-B,
Standard Model
below electroweak transition,
topological insulators,
triplet & singlet
chiral superconductor, ...



Bi_2Te_3

Present vacuum as semiconductor or insulator



electric charge of quantum vacuum

$$Q = \sum_a q_a = N\,[-1 + 3\times(-1/3) + 3\times(+2/3)\,] = 0$$

fully gapped 3+1 topological matter

superfluid ^{3}He-B, topological insulator Bi_2Te_3 , present vacuum of Standard Model

* Standard Model vacuum as topological insulator

Topological invariant protected by symmetry

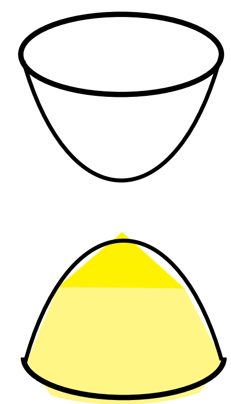
$$N_K = \frac{1}{24\pi^2}\, e_{\mu\nu\lambda}\, \mathbf{tr} \int dV\, K\, \mathbf{G}\, \boldsymbol{\partial}^\mu\, \mathbf{G}^{-1}\, \mathbf{G}\, \boldsymbol{\partial}^\nu\, \mathbf{G}^{-1} \mathbf{G}\, \boldsymbol{\partial}^\lambda\, \mathbf{G}^{-1}$$

over 3D momentum space

G is Green's function at ω=0, K is symmetry operator **GK** =+/− K**G**

Standard Model vacuum: $K=\gamma_5$ $\mathbf{G}\gamma_5 = -\gamma_5\mathbf{G}$

$$N_K = 8n_g$$



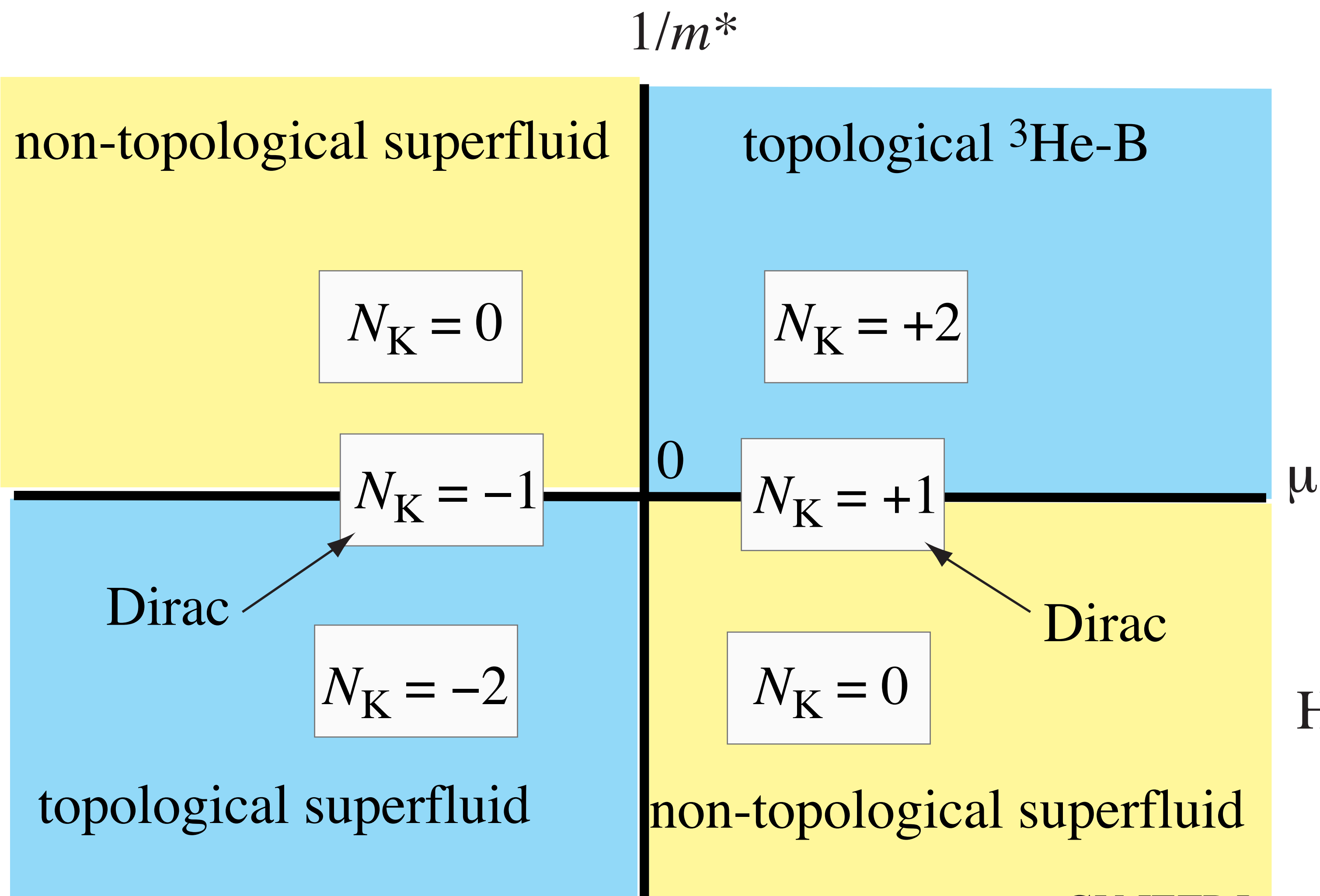
8 massive Dirac particles in one generation

topological superfluid ^{3}He-B

$$\mathrm{H}\tau_2 = -\tau_2\mathrm{H}$$

$$K = \tau_2$$

$$\mathrm{H} = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu \end{pmatrix} = \left(\frac{p^2}{2m^*} - \mu\right)\tau_3 + c_B \sigma\cdot\mathbf{p}\,\tau_1$$



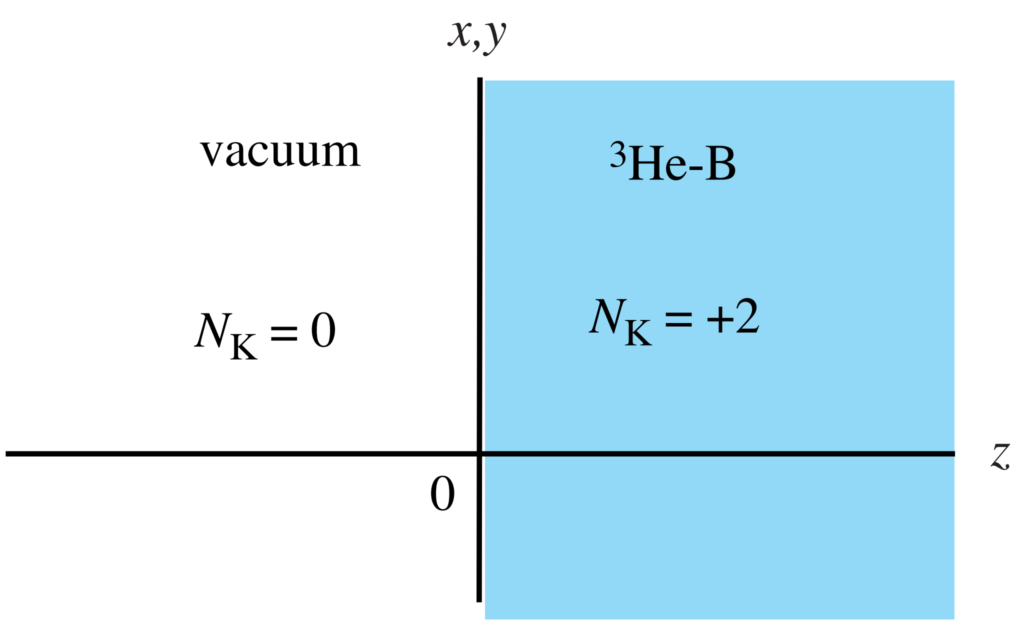
Dirac vacuum

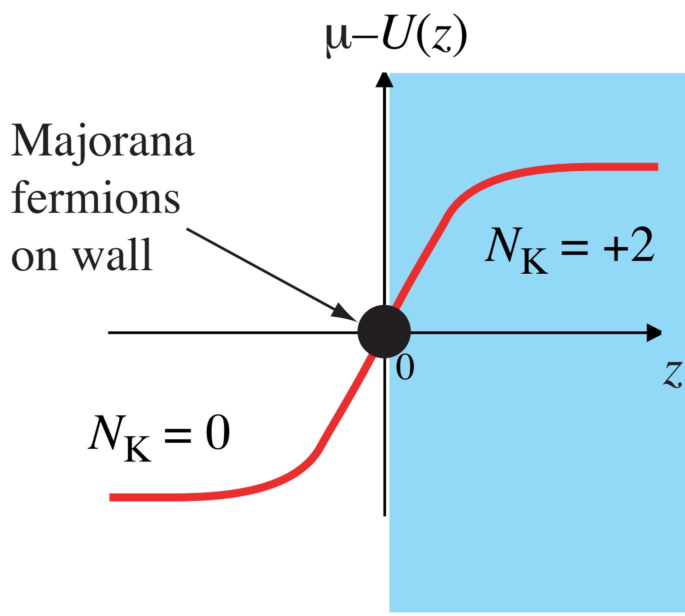
$$1/m^* = 0$$

$$\mathrm{H} = \begin{pmatrix} -M & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & +M \end{pmatrix}$$

GV JETP Lett. **90**, 587 (2009)

Boundary of 3D gapped topological superfluid





$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu + U(z) & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu - U(z) \end{pmatrix}$$

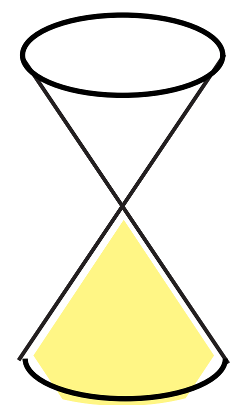
Majorana particle = Majorana anti-particle

1/2 of fermion: $b = b^+$

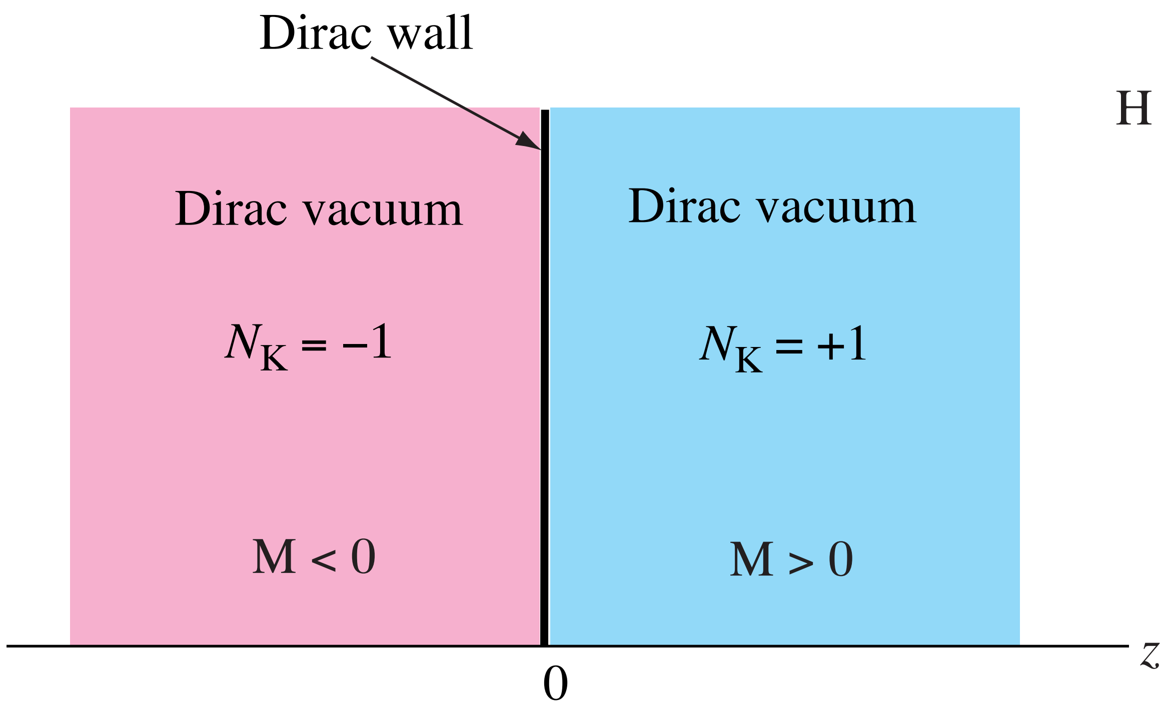
spectrum of Majorana zero modes

$$H_{zm} = c_B \hat{\mathbf{z}} \cdot \sigma \times \mathbf{p} = c_B (\sigma_x p_y - \sigma_y p_x)$$

helical fermions

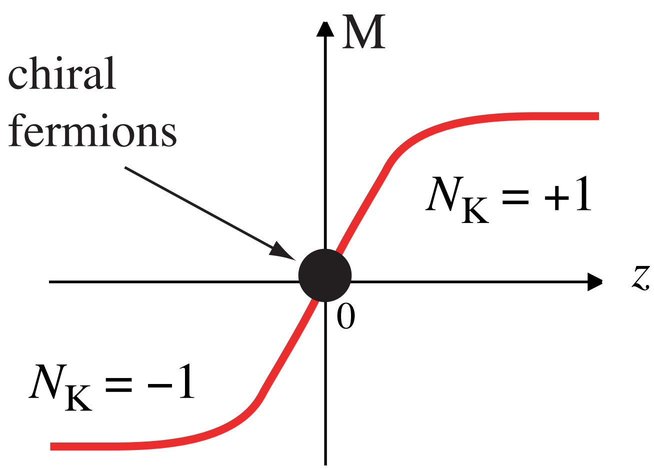


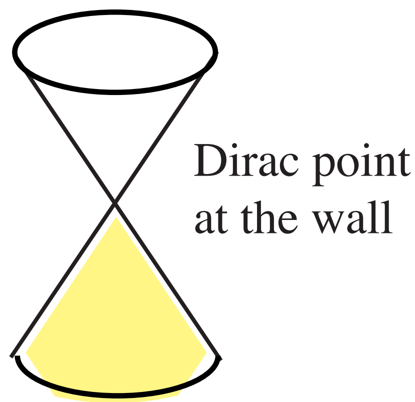
fermion zero modes on Dirac wall

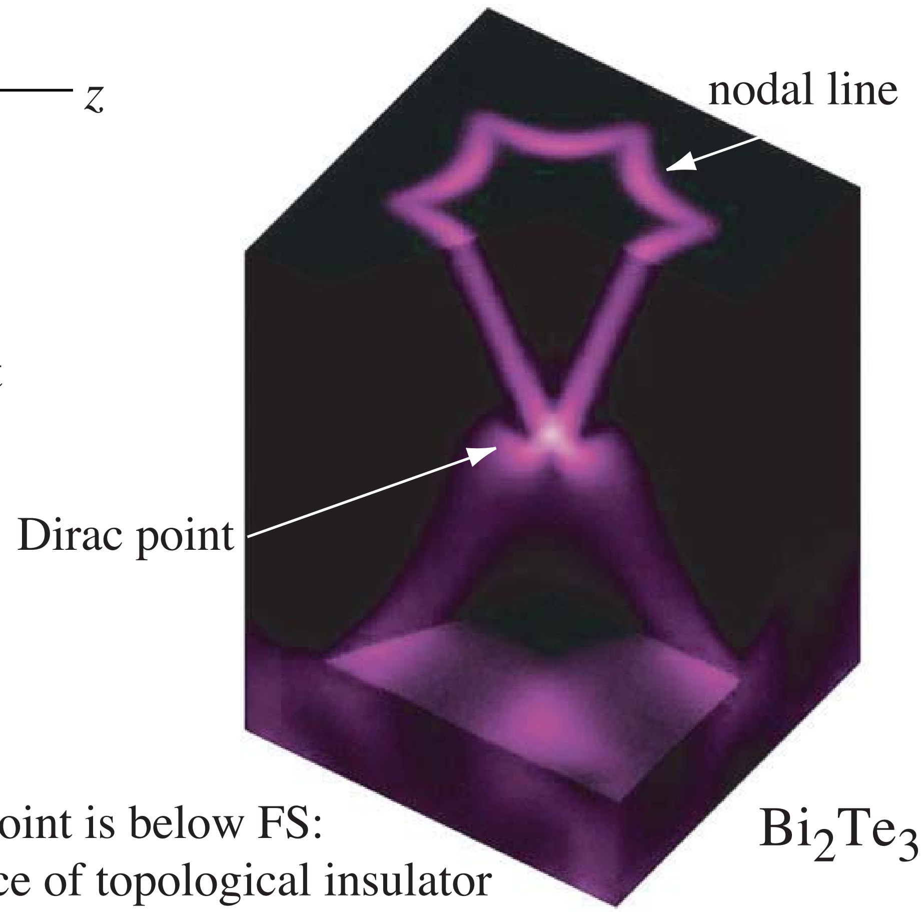


$$H = \begin{pmatrix} -M(z) & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & +M(z) \end{pmatrix}$$

Volkov-Pankratov,
2D massless fermions
in inverted contacts
JETP Lett. **42**, 178 (1985)





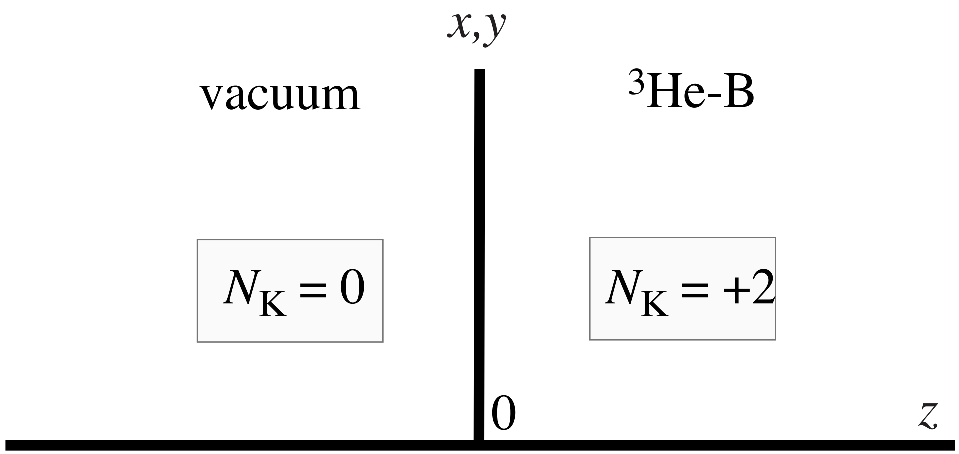


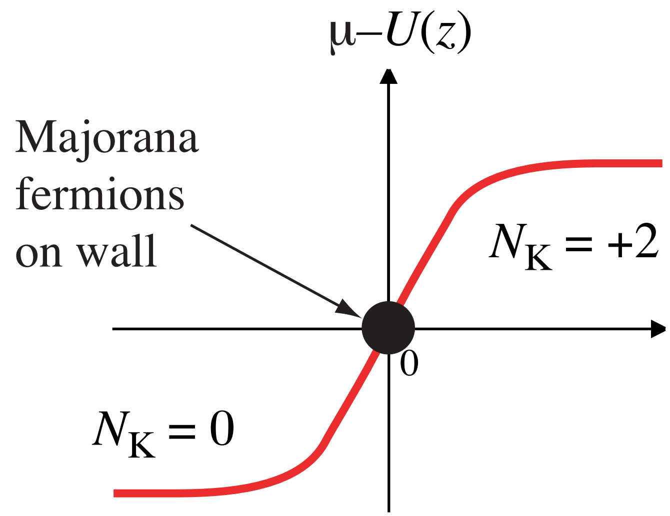
in Bi_2Te_3 Dirac point is below FS:
nodal line on surface of topological insulator

Majorana fermions: edge states on the boundary of 3D gapped topological matter

$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu + U(z) & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu - U(z) \end{pmatrix}$$

* boundary of topological superfluid ^{3}He-B



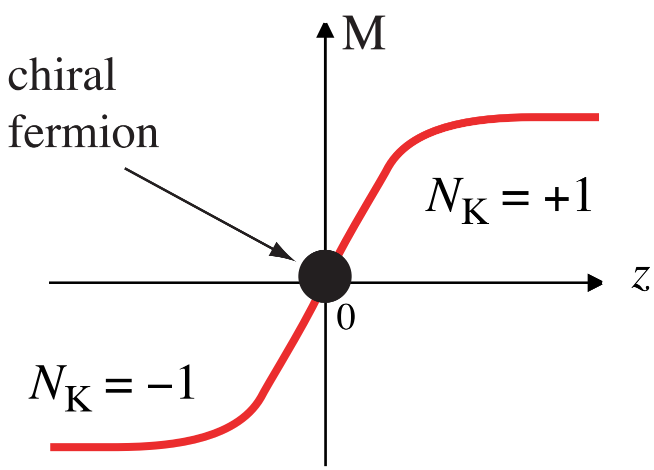


spectrum of fermion zero modes

$$H_{zm} = c\,(\sigma_x p_y - \sigma_y p_x)$$

helical fermions

* Dirac domain wall

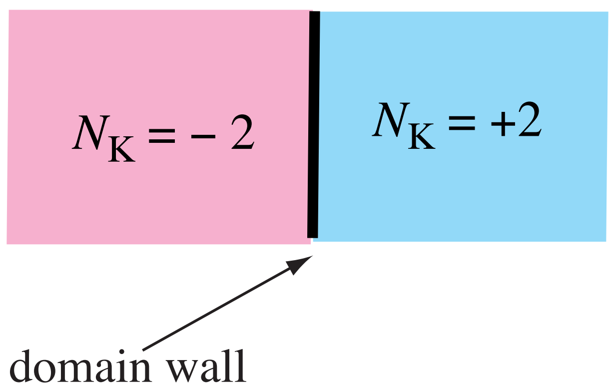


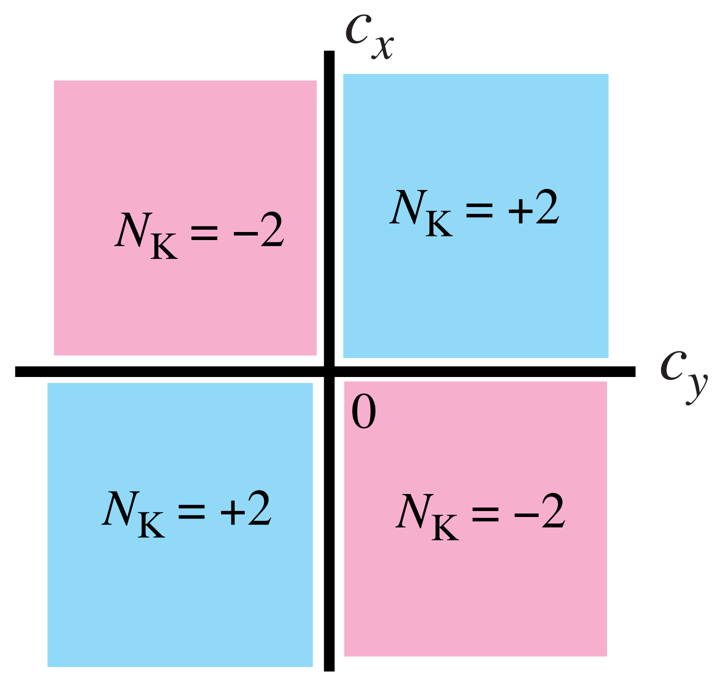
$$H = \begin{pmatrix} -M(z) & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & +M(z) \end{pmatrix}$$

Volkov-Pankratov,
2D massless fermions
in inverted contacts
JETP Lett. **42**, 178 (1985)

Majorana fermions on interface in topological superfluid ^{3}He-B

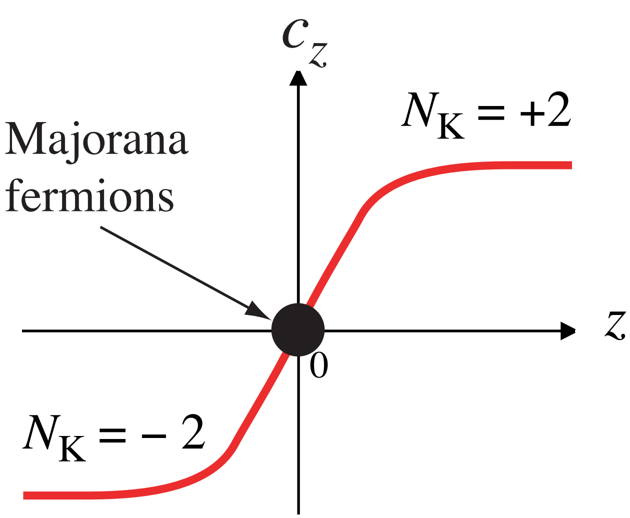
$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & \sigma_x c_x p_x + \sigma_y c_y p_y + \sigma_z c_z p_z \\ \sigma_x c_x p_x + \sigma_y c_y p_y + \sigma_z c_z p_z & -\frac{p^2}{2m^*} + \mu \end{pmatrix}$$

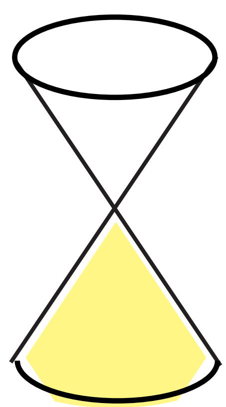




phase diagram

one of 3 "speeds of light" changes sign across wall





spectrum of fermion zero modes

$$H_{zm} = c\,(\sigma_x p_y - \sigma_y p_x)$$

Conclusion

Momentum-space topology determines:

universality classes of quantum vacua

effective field theories in these quantum vacua

topological quantum phase transitions (Lifshitz, plateau, etc.)

quantization of Hall and spin-Hall conductivity

topological Chern-Simons & Wess-Zumino terms

quantum statistics of topological objects

spectrum of edge states & fermion zero modes on walls & quantum vortices

chiral anomaly & vortex dynamics, etc.